Proof:

a. Assume $\sup \int_{[0,1]} |f_n|^2 dx \le M$, by Hölder's inequality, we have

$$\left|\int_{\{|f|\le N\}} f_n f\, dx\right| \le \left(\int_{\{|f|\le N\}} |f_n|^2 dx\right)^{\frac{1}{2}} \left(\int_{\{|f|\le N\}} |f|^2 dx\right)^{\frac{1}{2}} \le M^{\frac{1}{2}} \left(\int_{\{|f|\le N\}} |f|^2 dx\right)^{\frac{1}{2}}$$

let's now show $\int_{\{|f|\le N\}} f_n f\, dx \to \int_{\{|f|\le N\}} f^2 dx$ as $n\to\infty$

since $f_n \in L^2[0,1]$, $\int_{[0,1]} |f_n|\, dx \le \left(\int_{[0,1]} |f_n|^2 dx\right)^{\frac{1}{2}} \left(\int_{[0,1]} 1^2 dx\right)^{\frac{1}{2}} \le M^{\frac{1}{2}}$

$\exists\, S_n$ simple function such that $|f - S_m| \le \frac{1}{m}$, $|S_m| \le N$, $\forall x \in \{|f|\le N\}$

simply we can take $S_m = \sum_{k=-mN}^{mN-1} \frac{k}{m} \chi_{\{\frac{k}{m} \le f < \frac{k+1}{m}\}} + N \chi_{\{f=N\}} := \sum_k C_k \chi_{E_k}$

hence $\forall \varepsilon > 0$,

$$\left|\int_{\{|f|\le N\}} f^2 - f f_n\, dx\right| = \left|\int_{\{|f|\le N\}} (f^2 - f S_m) + (f S_m - f_n S_m) + (f_n S_m - f f_n)\, dx\right| \le$$

$$\int_{\{|f|\le N\}} |f(f-S_m)|\, dx + \left|\int_{\{|f|\le N\}} S_m (f - f_n)\, dx\right| + \int_{\{|f|\le N\}} |f_n (\ \ - f)|\, dx \le$$

$$\int_{\{|f|\le N\}} \frac{N}{m} dx + \sum_k \left| C_k \int_{\{|f|\le N\}} \chi_{E_k} (f - f_n)\, dx\right| + \frac{1}{m} \int_{\{|f|\le N\}} |f_n|\, dx \le$$

$$\frac{N}{m} + \sum_k N \left|\int_{\{|f|\le N\}\cap E_k} (f - f_n)\, dx\right| + \frac{M^{\frac{1}{2}}}{m} < \frac{\varepsilon}{3} + N \sum_k \left|\int_{\{|f|\le N\}\cap E_k} (f-f_n)\, dx\right| + \frac{\varepsilon}{3}$$

when m is sufficiently large, $< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$ when n sufficiently large $= \varepsilon$, therefore, let $n\to\infty$, we get

$$\left|\int_{\{|f|\le N\}} f^2 dx\right| \le M^{\frac{1}{2}} \left(\int_{\{|f|\le N\}} |f|^2 dx\right)^{\frac{1}{2}} \Rightarrow \int_{\{|f|\le N\}} |f|^2 dx \le M$$

since N is arbitrary, we have $\int_{[0,1]} |f|^2 dx = \int_{[0,1]} \lim_{N\to\infty} \chi_{\{|f|\le N\}} |f|^2 dx$

$= \lim_{N\to\infty} \int_{[0,1]} \chi_{\{|f|\le N\}} |f|^2 dx$ (Levi's theorem) $= \lim_{N\to\infty} \int_{\{|f|\le N\}} |f^2|\, dx \le M$

b. As in a. $\lim_{N\to\infty} \int_{\{|g|\le N\}} |g|^2 dx = \int_{[0,1]} |g|^2 dx$, thus

$\lim_{N\to\infty} \int_{\{|g|<N\}} |g|^2 dx = \int_{[0,1]} |g|^2 dx - \lim_{N\to\infty} \int_{\{|g|\ge N\}} |g|^2 dx = 0$

Also, from a. $\int_{[0,1]} |f - f_n|^2 dx \le \int_{[0,1]} (|f| + |f_n|)^2 dx = \int_{[0,1]} |f|^2 dx +$
$2\int_{[0,1]} |f f_n|\, dx + \int_{[0,1]} |f_n|^2 dx \le \|f\|_{L^2}^2 + 2M^{\frac{1}{2}} \|f\|_{L^2} + M$

we can find simple function t_m, such that $|g - t_m| \le \frac{1}{m}$, $|t_m| \le N$, $\forall x \in \{|g|\le N\}$

$t_m = \sum_k d_k \chi_{F_k}$, hence $\forall \varepsilon > 0$

$$\left|\int_{[0,1]} fg - f_n g\, dx\right| \le \left|\int_{\{|g|\le N\}} (f - f_n) g\, dx\right| + \left|\int_{\{|g|>N\}} (f-f_n) g\, dx\right| \le$$

$$\left|\int_{\{|g|\le N\}} (f-f_n)(g - t_m)\, dx\right| + \left|\int_{\{|g|\le N\}} (f-f_n) t_m\, dx\right| + \left(\int_{\{|g|>N\}} |f-f_n|^2 dx\right)^{\frac{1}{2}} \left(\int_{\{|g|>N\}} |g|^2 dx\right)^{\frac{1}{2}}$$

$$\le \left(\int_{\{|g|\le N\}} |f-f_n|^2 dx\right)^{\frac{1}{2}} \left(\int_{\{|g|\le N\}} |g-t_m|^2 dx\right)^{\frac{1}{2}} + \sum_k \left| d_k \int_{\{|g|\le N\}} \chi_{F_k} (f-f_n)\, dx\right| +$$

$$\|f - f_n\|_{L^2} \left(\int_{\{g>N\}} |g|^2 dx\right)^{\frac{1}{2}} < \frac{1}{m} \|f-f_n\|_{L^2} + N \sum_k \left|\int_{\{|g|\le N\}\cap F_k} (f-f_n)\, dx\right| + \frac{\varepsilon}{3}$$

when N is sufficiently large, $< \frac{\varepsilon}{3} + N \sum_k \left|\int_{\{|g|\le N\}\cap F_k} (f-f_n)\, dx\right| + \frac{\varepsilon}{3}$

when m is sufficiently large, $< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$, when n is sufficiently large

2011 August

Wednesday, December 26, 2018 21:08

1. (a) We should have $|X|=\sum_{i=1}^{r}|O_i|$, O_i are orbits, since $|G|=|G_s||O_i|$ where G_s is the corresponding stabilizer, $|O_i|=1,3,9$ or 27. then there are at least 2 O_i's such that $|O_i|=1$ since $|X|=50\equiv 2 \mod 3$

(b) $0\to A \xrightarrow{i} B \xrightarrow{f} C\to 0$ is exact, regard A as a subgroup of B then $C\cong B/\ker f = B/\operatorname{Im} i = B/A$, thus $A\trianglelefteq B$ and $|B|=[B:A]|A|=|B/A||A|=|A||C|=1350=2\times 3^3\times 5^2$, thus has a Sylow subgroup D of order 27, since $(27,50)=1$, $D\cap A=1$, $f(D)=C$

(c) If A is abelian, then $A\cong \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/5\mathbb{Z}\times\mathbb{Z}/5\mathbb{Z}$ or $A\cong\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/25\mathbb{Z}$, thus $H:=\mathbb{Z}/2\mathbb{Z}\leq A\trianglelefteq B$, let h denote 1 in $H=\mathbb{Z}/2\mathbb{Z}$, then $\forall b\in B$, $bhb^{-1}\in A$ with order 2 which are only be h itself, thus $1\neq h\in Z(B)$ is non-trivial

2. (a) Assume $a_1\in S=\{a_1,\cdots,a_n\}\subseteq F$ is not a root of unity, then $a_1^n\neq a_1^m$, $\forall n\neq m$, otherwise $a_1^{n-m}=1$, hence $a_1\mapsto a_1^m\mapsto a_1^{m^2}\mapsto\cdots$ would generate infinitely many different elements which is a contradiction

(b) Since $M\in GL_n(\mathbb{C})$, assume the characteristic polynomial of M is $(t-\lambda_1)\cdots(t-\lambda_n)$, then the characteristic polynomial of M^m is $(t-\lambda_1^m)\cdots(t-\lambda_n^m)$, since M^m are similar to M, they have the same characteristic polynomials, then apply result in (a). with $F=\mathbb{C}$, $S=\{\lambda_1,\cdots,\lambda_n\}$

3. $\forall\, x\frac{a}{b}\in xR_\pi\cap R$, then $b\mid x$, since $(b,\pi)=1$, $b\pi^r\mid x$, hence $x\frac{a}{b}=\pi^r\frac{x}{b\pi^r}a\in\pi^r R$, Conversely. $\forall\,\pi^r a\in\pi^r R$, $\pi^r a = x\frac{a}{\frac{x}{\pi^r}}\in xR_\pi\cap R$

4. (a) 1 get to be send to $\bigoplus_{i\in F}M_i$ for some finite subset F of I, so is the image of R

(b) Since $J\to\bigoplus_{i=1}^{\infty}J/J_i\hookrightarrow\bigoplus_{i=1}^{\infty}M_i$ and $\bigoplus_{i=1}^{\infty}M_i$ is injective, there is an extension $R\to\bigoplus_{i=1}^{\infty}M_i$, by (a), $\exists N$ such that $\bigoplus_{i\leq N}M_i$ contains the image of R, hence the image of J

(c) From (b), we know $\bigoplus_{i\leq N}J/J_i$ contains the image of J, thus $J_i=J$ $\forall i\geq N$ therefore R is Noetherian

5. (a) $x^{10}+\cdots+1$ is irreducible over $\mathbb{Q}[x]$, thus $1,\zeta,\cdots,\zeta^9$ are linearly independent over $\mathbb{Q}$, hence for any 10 elements of S could be transform into $1,\cdots,\zeta^9$ by multiplying ζ^k for some k

(b) $2\in(\mathbb{Z}/11\mathbb{Z})^\times$ is a generator corresponding to σ, then $\sigma(\alpha)=\zeta^2+\zeta^6+\zeta^8+\zeta^{10}+\zeta^7$ $\neq\alpha$ since they are linearly independent, thus $\alpha\notin\mathbb{Q}$

(c) Notice $\sigma^2(\alpha)=\alpha$, $|\operatorname{Aut}(\mathbb{Q}(\zeta)/\mathbb{Q}(\alpha))|=|\{1,\sigma^2,\sigma^4,\sigma^6,\sigma^8\}|=5$ by Galois Correspondence theorem, we have $[\mathbb{Q}(\alpha):\mathbb{Q}]=[\operatorname{Aut}(\mathbb{Q}(\zeta)/\mathbb{Q}):\operatorname{Aut}(\mathbb{Q}(\zeta)/\mathbb{Q}(\alpha))]$ $=2$

6. (a) Suppose ρ is not injective, denote $\ker\rho=H$, $\bar\rho: G/H\to GL_n(\mathbb{C})$ is a representation, but then G/H is abelian, $\bar\rho$ is completely reducible to a direct sum of representations of degree one, ρ is not irreducible which is a contradiction

(b) Consider $H=\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}-1&0\\0&-1\end{pmatrix},\begin{pmatrix}0&-1\\1&0\end{pmatrix},\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right\}$, $K=\{-1,1\}$, $G:=H\times K$ $\rho: G\to GL_2(\mathbb{C})$ via quotient map $H\times K\to H\times K/K\cong H\leq GL_2(\mathbb{C})$

Then this is a irreducible representation which is not injective

2012 August

Tuesday, January 1, 2019 11:39

1. (a) $\forall\, \sigma \in \mathrm{Aut}(N)$, $\sigma(aba^{-1}b^{-1}) = \sigma(a)\sigma(b)\sigma(a)^{-1}\sigma(b)^{-1} \in N'$, hence N' is a characteristic subgroup of N, thus $N' \trianglelefteq G$

(b) Since G is solvable, so is N, consider $1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_{r-1} \trianglelefteq N_r = N$ with N_{i+1}/N_i abelian, thus $N' \leq N_{r-1}$, but $N' \trianglelefteq G$ and N is minimal, hence N' can only be 1, thus $N = N/N'$ is abelian

(c) Since N is abelian, finite and minimal, $|N| = p^r$, and $N \cong (\mathbb{Z}/p\mathbb{Z})^r$ otherwise, N would have some nontrivial proper characteristic subgroup which would be a normal subgroup of G, contradicts the minimality of N

2. Take the basis of V such that the matrix of T is of Jordan canonical form, consider the following simple case $T = \begin{pmatrix} \lambda & 1 & \\ & \ddots & \ddots \\ & & \lambda \end{pmatrix}$, then

$\mathrm{rank}(T) = \begin{cases} n, & \lambda \neq 0 \\ n-1, & \lambda = 0 \end{cases}$, $\mathrm{rank}(I-T) = \begin{cases} n, & \lambda \neq 1 \\ n-1, & \lambda = 1 \end{cases}$, hence $\mathrm{rank}(T) + \mathrm{rank}(I-T)$ $\begin{cases} 2n, & \text{else} \\ 2n-1, & \lambda = 0 \text{ or } 1 \end{cases}$ thus $\mathrm{rank}(T) + \mathrm{rank}(I-T) = n$ iff $n = 1$, $\lambda = 0$ or 1, therefore, $\mathrm{rank}(T) + \mathrm{rank}(I-T) \geq \dim V$, equality holds iff $T = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ then $T^2 = T$

3. (a) Irreducible elements are prime elements in a UFD

(b) $0 \neq P \subsetneq R$ is a prime ideal, then $\exists\, 0 \neq a \in P$, $a = \pi_1 \cdots \pi_k$ a product of irreducible elements, then one of π_k and $\pi_1 \cdots \pi_{k-1}$ must belongs to P, by induction we know P contains a prime ideal of the form πR for some irreducible π.

(c) Suppose $m \subsetneq R$ is a nonzero maximal ideal, then it's also prime, by (b), $\pi R \subseteq m$ for some irreducible π, but then πR is a prime thus maximal ideal, hence $m = \pi R$. namely every maximal ideal is a principle ideal. Since $I := \langle S \rangle \subsetneq R$, $I \subseteq m = \pi R$ for some maximal ideal, thus there is an irreducible π such that π divides r_i for all i

(d) $0 \neq I \subsetneq R$ is an ideal, suppose $S := \{r_i\}_{i \in I}$ is a set of generators, then by (c), $\exists\, \pi$ divides r_i, then let $S' := \{r_i' = \frac{r_i}{\pi}\}_{i \in I}$, $\langle S \rangle = \pi \langle S' \rangle$ continue doing so until one of r_i becomes 1, thus $\langle S \rangle = \pi^k R$ which is principle

4. (a) $GF(x) = G(f_1(x), \alpha(x)) = \beta f_1(x) - f_2 \alpha(x) = 0$. Conversely, if $(p,k) \in \mathrm{Ker}\, G$, then $g_2 \beta(p) = g_2 f_2(k) = 0$, hence $\mathrm{id}_M\, g_1(p) = g_2 \beta(p) = 0 \Rightarrow g_1(p) = 0 \Rightarrow \exists\, x \in K_1$, such that $f_1(x) = p$, then $f_2 \alpha(x) = \beta f_1(x) = \beta(p) = f_2(k)$, since f_2 is injective, $\alpha(x) = k$, hence $F(x) = (f_1(x), \alpha(x)) = (p, k)$, $0 \to K_1 \xrightarrow{F} P_1 \oplus K_2 \xrightarrow{G} P_2 \to 0$ is exact

(b) Since P_2 is projective, $K_2 \oplus P_1 \cong K_1 \oplus P_2$, namely the exact sequence splits

(c) Suppose $A \oplus B = \langle (a_1, b_1), \cdots, (a_n, b_n) \rangle$, then $\forall\, a \in A$, $(a, 0) = r_1(a_1, b_1) + \cdots + r_n(a_n, b_n)$ $= (r_1 a_1 + \cdots + r_n a_n, r_1 b_1 + \cdots + r_n b_n) \Rightarrow a = r_1 a_1 + \cdots + r_n a_n$, thus $A = \langle a_1, \cdots, a_n \rangle$

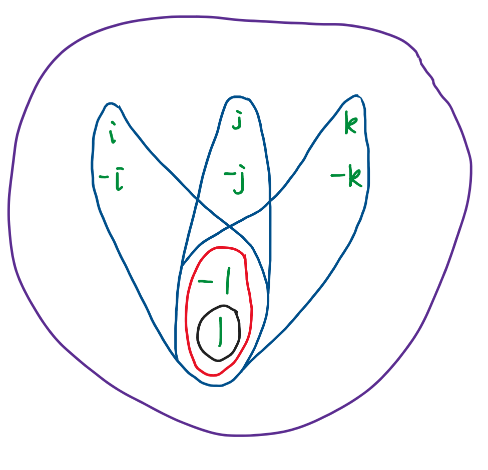
(d) There should be a commutative diagram involving two exact sequences

$$\begin{array}{ccccccccc} 0 & \to & K & \to & P & \to & R/I & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & I & \to & R & \to & R/I & \to & 0 \end{array}$$

where R is evidently finitely generated and free hence projective

If both K and P are finitely generated and P is projective, then by (b) and (c) $P \oplus I \cong K \oplus R$ is finitely generated, hence so should I which is a contradiction

5. You can draw a picture easily showing the subgroups



(a) Subextensions $F/\mathbb{Q}$ such that $[F:\mathbb{Q}]$ corresponds to subgroups of order 4

(b) c corresponds to -1 since it's the only element of order 2 and $\bar{z} = z\ \forall z \in \mathbb{C}$, thus we can take σ to be i

(c) By (a) we know that $\mathrm{Aut}(L/F) = \{1, -1, i, -i\}$ or $\{1, -1, j, -j\}$ or $\{1, -1, k, -k\}$ hence c acts trivially on F

6. In order to decide how many trivial representations $\tilde{\rho}$ contains. We need to decide the dimension of $\bigcap_{g \in G} \mathrm{Ker}\left(M - \rho(g) M \rho(g)^{-1}\right) = \bigcap_{g \in G} \mathrm{Ker}\left(M\rho(g) - \rho(g) M\right)$

Since $\rho(g)^k = 1$, thus each $\rho(g)$ is diagonolizable,

And $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \lambda & \xi \\ \eta & \mu \end{pmatrix} - \begin{pmatrix} \lambda & \xi \\ \eta & \mu \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} b\eta - c\xi & \xi(a-d) - b(\lambda - \mu) \\ c(\lambda - \mu) - \eta(a-d) & b\eta - c\xi \end{pmatrix} = 0 \Leftrightarrow$

$\begin{cases} b\eta = c\xi \\ b(\lambda - \mu) = \xi(a-d) \\ c(\lambda - \mu) = \eta(a-d) \end{cases}$

(a) If ρ is irreducible, then find a basis such that $\rho(g) = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, $\lambda \neq \mu$, for some g, we get $b = c = 0$, but since ρ is irreducible, on this basis, $\exists\, g \neq h \in G$, such that h is not a diagonal matrix, which means ξ, η are not both zeros, hence $a = d$, thus $\tilde{\rho}$ contains the trivial representation exactly once.

(b) If ρ is the sum of two distinct one-dimensional representations, then on some basis and for some g, $\rho(g) = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ and $\rho(h)$, $\forall h \in G$ are all diagonal matrices, by analysis in (a), we know that $\tilde{\rho}$ contains the trivial representation exactly twice

(c) If ρ is the sum of two equal one-dimensional representations, then on any basis, $\forall g \in G$, $\rho(g)$ are scalar matrices, thus $\tilde{\rho}$ equals the sum of four copies of the trivial representation

2012 January

Saturday, January 12, 2019 12:38

2. For any $A \in S$, since A is nilpotent, $\operatorname{Ker} A \neq 0$, but if $\operatorname{Ker} A = \mathbb{C}^n$, then $A = 0$ which can be ignored, pick $A \neq 0$, then since $AB = BA$, $\forall A, B \in S$, we have $A(B(\operatorname{Ker} A)) = B(A(\operatorname{Ker} A)) = 0 \Rightarrow B(\operatorname{Ker} A) \subseteq \operatorname{Ker} A$, $\forall B \in S$, thus find a basis for $\operatorname{Ker} A$, then $B = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix}$, $\forall B \in S$, since

$$BC = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix} \begin{pmatrix} C_1 & C_2 \\ 0 & C_3 \end{pmatrix} = \begin{pmatrix} B_1 C_1 & B_1 C_2 + B_2 C_3 \\ 0 & B_3 C_3 \end{pmatrix} = \begin{pmatrix} C_1 B_1 & C_1 B_2 + C_2 B_3 \\ 0 & C_3 B_3 \end{pmatrix} = CB, \quad \begin{cases} B_1 C_1 = C_1 B_1 \\ B_3 C_3 = C_3 B_3 \end{cases}$$

similarly we have B_1, B_3 are nilpontent

Then we should use indution to continue this process

2014 August

Wednesday, January 2, 2019 17:00

1. Since $N \not\subset Z(G)$ and $|N|=3$, $N\cap Z(G)=1$, consider the group action by conjugation, since N is a normal subgroup, assume $N=\{1,\alpha,\alpha^2\}$, then $g\alpha g^{-1}=\alpha$ or α^2, if $\forall g\in G$, $g\alpha g^{-1}=\alpha$, then $\alpha\in Z(G)$ which is impossible, thus $\{\alpha,\alpha^2\}$ is an orbit, then the stabilizer of α H is a subgroup such that $[G:H]=2$

2. (a) Since $\operatorname{rank}(A)=1$, $A\vec{1}=n\vec{1}$, thus $ch_A(x)=x^{n-1}(x-n)$

(b) $m_A(x) \mid x(x-n)$, hence $m_A(x)=x(x-n)$, thus the Jordan normal form is $\begin{pmatrix} n & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0\end{pmatrix}$

3. (a) $\frac{a}{b}\otimes\frac{c}{d}=\left(d\frac{a}{bd}\right)\otimes\left(c\frac{1}{d}\right)=cd\left(\frac{a}{bd}\right)\otimes\frac{1}{d}=\left(c\frac{a}{bd}\right)\otimes\left(d\frac{1}{d}\right)=\frac{ac}{bd}\otimes 1$

(b) $\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Q}\cong\mathbb{Q}\otimes_{\mathbb{Z}}1\cong\mathbb{Q}$

4. Let $\Sigma=\{I\subseteq R \text{ ideal} \mid I \text{ doesn't contain any product of prime ideals}\}$, Suppose Σ is not empty, since R is Noetherian, Σ has a maximal element J, evidently J itself can't be prime, by suppose $x,y\notin J$, then $J\subsetneq J+(x)$ contains a product of primes, so is $J+(y)$, by then so does $J+(xy)=(J+(x))(J+(y))$, hence $xy\notin J$. namely J is prime which is a contradiction

5. (a) Suppose $2\le \deg f\le 4$, then consider A_5 acts on its roots, thus there is a homomorphism $A_5\to S_n$ where $n=\deg f$, since $60=|A_5|>|S_n|$, thus the Kernel will be a nontrivial normal subgroup of A_5 which then can only be A_5 itself since A_5 is a simple group, but then A_5 acts trivially on roots of f which is a contradiction

(b) Consider the subgroup H generated by $(1\ 2\ 3)$, then $|H|=3$, hence the intermediate field F fixed by H satisfies $[K:F]=|H|$, thus $[F:\mathbb{Q}]=20$, since $\mathbb{Q}$ is a characteristic 0, there is a primitive element $\alpha\in F$ such that $F=\mathbb{Q}(\alpha)$ then the minimal polynomial $g(x)\in\mathbb{Q}[x]$ is of degree 20 and irreducible

6. (a) Normal subgroup should consist of conjugacy classes, try the computation would show that G is simple, but since G has irreducible representations of degree >1, thus G is not abelian, hence G is not solvable

Remark: The number of irreducible degree 1 complex representations is $1=|G/G'|$, hence $G=G'$, G can't be solvable

(b) Consider the irreducible representations of G of deg 3, shows there is a homomorphism from $G\to GL_3(\mathbb{C})$ which is not trivial, thus must be injective

Remark: From χ_2 or χ_3, we know the corresponding representations are injective

(c) Notice any two subgroups of order 3 intersects at identity, hence the number of order 3 elements must be even, from χ_2 we see that $\xi_1+\xi_2+\xi_2\ne\frac{-1\pm\sqrt{7}i}{2}$, $\forall \xi_1,\xi_2,\xi_3\in\{1,\omega,\omega^2\}$, thus the number of order 3 elements could only be 42, the number of or order 3 subgroups is then 21

2014 January

Tuesday, January 1, 2019 18:34

1.(a) Let $|K|=j$, then $|G|:=n=ij$, $(i,j)=1$, since K is normal, $|G/K|=i$, hence $\forall x\in G$, $\overline{x^i}=\bar{x}^i=1\Rightarrow x^i\in K$. Also, since $(i,j)=1$, $ai+bj=1$ thus $\forall x\in K$, $1=x^{bj}=x^{1-ai}\Rightarrow x=x^{ai}$, therefore $K=\{x^i\mid x\in G\}$

(b) Consider $G=S_3$, $K=\{id, (1\ 2)\}$ which is not a normal subgroup, $|K|=2$, $[G:K]=3$ are relatively prime, but $(2\ 3)^3=(2\ 3)\notin K$

2. Since $T^2=T$, its Jordan conanical form should be $\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}$, but then $I-\alpha T=\begin{pmatrix}(1-\alpha)I & 0\\ 0 & I\end{pmatrix}$ which is invertible provided $\alpha\neq 1$

3.(a) Consider $(x^2+1)(x^4-x^2+1)=x^6+1$ in $\mathbb{Z}[X]/I$ shows that $\mathbb{Z}[X]/I$ is not an integral domain, hence I is not prime

(b) Consider $B=(3,X)$, $A=(3)$, since 3 is an irreducible element in $\mathbb{Z}[X]$, A is prime, $\mathbb{Z}[X]/B\cong\mathbb{Z}/3\mathbb{Z}$ which is a field, B is maximal hence prime, also $A\subset I\subset B\subset\mathbb{Z}[X]$

4. Let $F=Re_1+\cdots+Re_n$ be a finitely generated R-module, if F is flat, then $\forall I\subseteq R$, $0\to I\hookrightarrow R$ is exact, so is $0\to I\otimes_R F\hookrightarrow R\otimes_R F$, but $I\otimes_R F\cong IF$, $R\otimes_R F\cong F$, hence $IF\hookrightarrow F$ is an inclusion, now suppose $r\in R$ is not a zero divisor and $rf=0$ for $f\in F$, Let $I=Rr$, we know that $f=0$ which shows that F is torsion free. Conversely, if F is torsion free, $R\cong Re_i$, hence $\forall M\hookrightarrow N$, $M\otimes Re_i\cong M\otimes R\cong M\hookrightarrow N\cong N\otimes R\cong N\otimes Re_i$, thus $M\otimes F=M\otimes(Re_1+\cdots+Re_n)$ $\cong M\otimes Re_1+\cdots+M\otimes Re_n\hookrightarrow N\otimes Re_1+\cdots+N\otimes Re_n\cong N\otimes(Re_1+\cdots+Re_n)=N\otimes F$

Thus F is flat

5.(a) Since $K/\mathbb{Q}$ is Galois, $|G|=[K:\mathbb{Q}]=21$, and G is non-abelian, use Sylow's theorems, we know that G has only one normal subgroup of order 7, and 7 subgroups of order 3, it is obvious that the splitting field of $f(x)$ denoted by L is contained in K, and since f is irreducible $[L:\mathbb{Q}]\geq 7$, thus $H:=Gal(K/L)\trianglelefteq G$ with $|H|\leq 3$, thus H can only be the trivial group and $L=K$

(b) According to Sylow's theorems, we know that all 7 subgroups of order 3 are conjugate to each other, hence there are 7 distinct fields F_i, $1\leq i\leq 7$ with $\mathbb{Q}\leq F_i\leq K$, $[F_i:\mathbb{Q}]=7$, let $H_i=Gal(K/F_i)$, then $\exists g\in G$ such that $g^{-1}H_1g=H_2$ thus $\forall\sigma\in G$, such that fixes $g(F_1)$ means $g\sigma g^{-1}$ fixes $F_1\Leftrightarrow g\sigma g^{-1}\in H_1\Leftrightarrow$ $\sigma\in g^{-1}H_1g=H_2$, hence $g(F_1)=F_2$

(c) Since $\mathbb{Q}(\alpha_i)\cong\mathbb{Q}[X]/(f(X))$, $[\mathbb{Q}(\alpha_i):\mathbb{Q}]=7$, consider G acting on $\{\alpha_1,\cdots,\alpha_7\}$ trasitively, with stabilizer of order 3, thus $Gal(K/\mathbb{Q}(\alpha_i))$ corresponds to those subgroups of order 3, and thus $\mathbb{Q}(\alpha_i)$ are exactly F_i's

6.(a) Consider $T: G\to G$ by sending elements to its inverses, which is a permutation of order 2, $T^{-1}=T$, hence $\forall r=\sum\beta_i g_i\in R$, we have $\rho_2(g)\circ T(r)$ $=\sum\beta_i\rho_2(g)\circ T(g_i)=\sum\beta_i\rho_2(g)(g_i^{-1})=\sum\beta_i gg_i^{-1}$, and on the other hand $T\circ\rho_1(g)(r)=$ $\sum\beta_i T\circ\rho_1(g)(g_i)=\sum\beta_i T(g_ig^{-1})=\sum\beta_i gg_i^{-1}$, thus ρ_2 and ρ_3 are equivalent

(b) $\rho_1(g)$ always takes elements of G to its conjugates, particularly, $\mathbb{C}1$ will be a G-invariant subspace, $\rho_2(g)$ act on G as permutations, thus always irreducible, but ρ_1 is irreducible only if G is trivial. Conversely, if G is trivial, ρ_1 and ρ_2 are both trivial hence equivalent

(c) Consider $\varphi: G\to G$, $g\mapsto g^2$ which is a homomorphism since G is abelian If ρ_2 and ρ_4 are equivalent, and ρ_2 is faithful, so is ρ_4, but if G has even order, then $\rho_4=\rho_2\circ\varphi$ won't be faithful. Conversely, If G has odd order, then φ would be an isomorphism since 2 and $|G|$ are relatively prime. Hence ρ_4 and ρ_2 are both irreducible and of degree $|G|$, hence they are equivalent

2015 August

Friday, January 4, 2019 16:56

1\. (a) $|B| = 85 \times 9 = 5 \times 17 \times 3^2$, thus B has a subgroup of order 9, denoted H, then $H \cap A = 1$, $g(H) = C$

(b) By Sylow's theorems, $A \cong \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/17\mathbb{Z}$, thus $|\mathrm{Aut}(A)| = 64$, by sending $(1,0)$ to $(n,0)$, $1 \leq n \leq 4$, and $(0,1)$ to $(0,m)$, $1 \leq m \leq 16$ independently

(c) Consider B acts on A by conjugation, which would induce a homomorphism $B \to \mathrm{Aut}(A)$, and A is in the kernel since it's abelian, thus B is the kernel, hence $A \leq Z(B)$, thus elements of A and H commutes, hence $B \cong A \times H$ thus abelian since H is also abelian

2\. (a) Notice $M^n = I$, and $x^n - 1 = 0$ has n different roots, thus its characteristic and minimal polynomials are both $x^n - 1$

(b) Let $N = J + K$, Where $J = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{pmatrix}$, $K = \begin{pmatrix} 0 & \cdots & & 1 \\ & \ddots & & \vdots \\ & & & 0 \end{pmatrix}$, then $JK = KJ = K^2 = 0$, thus $(J+K)^n = J^n$, hence $N^n = 0$ and $N^{n-1} \neq 0$, therefore, its characteristic and minimal polynomials are both x^n

3\. (a) Notice $\mathbb{C} = \mathbb{C}/M$ is a R-module, verify that $M \times M \to \mathbb{C}$ sending (f,g) to $\frac{\partial f}{\partial X}(0,0)\frac{\partial g}{\partial Y}(0,0)$ is an R-bilinear map, thus induce $M \otimes_R M \to \mathbb{C}$ an R-homomorphism which is also the well-defined map of vector spaces over $\mathbb{C}$

(b) Consider the image of Z in $M \otimes_R M \to \mathbb{C}$ which would be $1 \neq 0$ thus $Z \neq 0$ in $M \otimes_R M$

(c) Consider $r = XY \neq 0$ in R, but $rZ = (XY)(X \otimes Y - Y \otimes X)$
$= (XY) \otimes (XY) - (XY) \otimes (XY) = 0$

4\. (a) We only need to show that if $I \cap J \subseteq P$, so is one of I or J. Suppose not, $\exists i \in I \setminus P$, $j \in J \setminus P$. but $ij \in IJ \subseteq I \cap J \subseteq P$ which is a contradiction

(c) Suppose there is another minimal prime ideal P, then $m_1 \cap \cdots \cap m_n = (0) \subseteq P$ which by (a) implies one of $m_i \subseteq P$ since P is minimal, $P = m_i$

5\. Since each extension K_{i+1}/K_i is Galois and of degree 3, thus $K_n/\mathbb{Q}$ is also Galois and of degree of a power of 3, but if $\mathbb{Q}(\sqrt[3]{2}) \subseteq K_n$, then K_n contains the splitting field of $\sqrt[3]{2}$ denoted by L, but then $L/\mathbb{Q}$ is Galois but of degree 6 which leads to a contradiction

$K_n/\mathbb{Q}$ is of degree of a power of 3

2015 January

Friday, January 4, 2019 11:51

1. (a) By Sylow's theorems, since H is not normal, H can only have exactly q conjugates including itself

(b) Consider G acts on H by conjugation induces a homomorphism $\phi: G \to S_q$. $\operatorname{Ker}\phi = \cap$ stabilizers $\leq H$, and $\operatorname{Ker}\phi \neq H$. $|H| = p^2$, $|\operatorname{Ker}\phi| = 1$ or p, thus pq divides $|\operatorname{Im}\phi|$

2. (a) Suppose the characteristic polynomial $ch(t) = f(t)g(t)$ is reducible over F, then we know $\{v, Tv, \cdots, T^{n-1}v\}$ is a basis of V for any $v \neq 0$ assuming $\dim V = n$, then for some $v \neq 0$ we have $w := g(T)v \neq 0$, but then $f(T)w = f(T)g(T)v = ch(T)v = 0$ which is a contradiction

(b) Suppose $v \neq 0$ is not a cyclic vector for T, namely, $Tv = \lambda v$ for some $\lambda \in F$ which is a contradiction since $ch(t)$ is irreducible

3. If $0 = f^N = a_0^N + g(x)X$, $g \in R[[X]] \Rightarrow a_0^{N_0} = 0$ then we have $(f - a_0)^{2N_0} = 0$ since in the expansion every term contains a power of either f or a_0 which $\geq N_0$, thus $(f-a_0)^{2N_0} = X^{2N_0} h^{2N_0} = 0 \Rightarrow h^{2N_0} = 0 \Rightarrow a_1^{2N_0} = 0$, thus make an induction.

Conversely, If $a_n^{N_n} = 0$, then consider ideals $I_n = (a_0, \cdots, a_n)$, $I_0 \subseteq \cdots \subseteq I_n$ is an accending chain, thus $I_k = I_{k+1} = \cdots$, hence $a_{k+1}, a_{k+2}, \cdots$ can be generated by $a_0, \cdots, a_k$. suppose $a_0^M = \cdots = a_k^M = 0$, $a_j = r_{j0}a_0 + \cdots + r_{jk}a_k$. Let $N = (k+1)M$ then coefficients of the monomials in the expansion has a power at least of order M on one of $\{a_0, \cdots, a_k\}$, thus $f^N = 0$

4. Suppose $\{v_1, \cdots, v_n\}$, $\{w_1, \cdots, w_m\}$ are base for V and W correspondingly, and let $\{v_1^*, \cdots, v_n^*\}$ be a dual basis such that $\langle v_i, v_j^* \rangle = v_j^*(v_i) = \delta_{ij}$, let $\xi_{ij} \in \operatorname{Hom}(V^*, W)$ such that $\xi_{ij}(v_k^*) = \delta_{ik} w_j$, then ξ_{ij} form a basis for $\operatorname{Hom}(V^*, W)$, since $\dim V \otimes W = \dim \operatorname{Hom}(V^*, W) = nm$, we only need to show that ϕ is surjective, but notice $\phi(v_i \otimes w_j)(v_k^*) = v_k^*(v_i)w_j = \delta_{ik}w_j$, thus $\phi(v_i \otimes w_j) = \xi_{ij}$

5. (a) Notice that $u - v = (r_1 - r_4)(r_3 - r_2)$, since f is irreducible over $\mathbb{Q}$ which has characteristic 0, r_1, r_2, r_3, r_4 must be distinct, thus $u \neq v$

(b) $g(x)$ is invariant under S_4, hence coefficients of g are symmetric in r_1, r_2, r_3, r_4, hence $\in \mathbb{Q}$

(c) $u, v, w \notin \mathbb{Q}$, otherwise it would be invariant under S_4, then some two of them must be equal which contradicts (a). subsequently $g(x)$ is irreducible, otherwise would have some as root in $\mathbb{Q}$

(d) Consider $(1\ 2), (1\ 2\ 3) \in S_4$ induce $(u\ v), (u\ w\ v) \in S_3$, and $\operatorname{Gal}(\mathbb{Q}(u,v,w)/\mathbb{Q}) \leq S_3$, hence $\operatorname{Gal}(\mathbb{Q}(u,v,w)/\mathbb{Q}) = S_3$

6. (a) $1^2 + 1^2 + 1^2 + a^2 = 12 \Rightarrow a = 3$, Assume $|C_2| = r_2$, $|C_3| = r_3$, $|C_4| = r_4$, then from the orthogonal relation we know that

$$\begin{cases} 1 + r_2 + r_3\omega^2 + r_4\omega = 0 \\ 1 + r_2 + r_3\omega + r_4\omega^2 = 0 \end{cases} \Rightarrow r_3 = r_4,$$

on the other hand, from the class equation we can deduce that $r_2 = 3$, $r_3 = r_4 = 4$, then use orthogonal relation again, we have

$$\begin{cases} 3 + 3b + 4c + 4d = 0 \\ 3 + 3b + 4c\omega^2 + 4d\omega = 0 \\ 3 + 3b + 4c\omega + 4d\omega^2 = 0 \end{cases} \Rightarrow \begin{cases} b = -1 \\ c = 0 \\ d = 0 \end{cases}$$

(b) Since G has has only one orbit of size 3, hence it has a normal subgroup of order 4

(c) If G has a normal subgroup of order 2, then G should have two orbits of size 1 which is a contradiction

2016 August

Sunday, January 13, 2019 22:41

5. (a) Suppose $f(x) = (x-\alpha_1)\cdots(x-\alpha_m)(x-\beta_1)\cdots(x-\beta_n) = hg$, let $S_1 = \{\alpha_1, \cdots, \alpha_m\}$, $S_2 = \{\beta_1, \cdots, \beta_n\}$, $S = S_1 \cup S_2$, $L = K(S)$, $G = Aut(L/\mathbb{Q})$, $H = Aut(L/K)$, $G/H \cong Aut(K/\mathbb{Q})$

Since f is irreducible over $\mathbb{Q}$, g, h are irreducible over K, $\exists\, \rho \in G$ such that $\rho(\alpha_1) = \beta_1$, hence $0 = \rho(h(\alpha_1)) = \rho(h)(\beta_1) \Rightarrow g \mid \rho(h) \Rightarrow \deg g \leq \deg h$, similary consider ρ^{-1} we have $\deg h \leq \deg g$, thus $g = \rho(h)$, H acts on h trivially, thus induce $\sigma = \rho H \in G/H \cong Aut(K/\mathbb{Q})$, Therefore we have $g = \sigma(h)$

(b) Consider $K = \mathbb{Q}(\sqrt[3]{2})$, $K/\mathbb{Q}$ is not Galois, $f = x^3 - 2 = (x - \sqrt[3]{2})(x^2 + \sqrt[3]{2}x + \sqrt[3]{4})$ $= hg$ which have different degrees