I. The groups K0 and K1

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Grothendieck introduced the group $K_0(R)$ of a commutative ring R as the free abelian group on isomorphism classes of fin. gen. projective R-modules modulo the subgroup generated by elements of the form $[M]-[M']-[M'']$ for every exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of R-modules. Here $[M]=[M'\oplus M'']$ as M'' is projective. $K_0(R)$ becomes a ring by $[M]\cdot[M']=[M\otimes M']$ as tensor preserves exact sequences of projective modules.

<u>Examples</u>: (1) Let R is a field or more generally local ring

→ f.g. proj. modules are free over local rings

→ $rk: K_0(X) \xrightarrow{\sim} \mathbb{Z}$

$$[R^n] \longmapsto n$$

is a ring isomorphism

(2) Let R Dedekind domain.

→ $rk(M) := rk(M\otimes R_p)$ is independent of p

→ fg projective modules are sums

$$M = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r \simeq R^{r-1} \oplus \mathcal{L}_1 \cdots \mathcal{L}_r$$

of rank 1 projective modules

→ rank 1 projective modules are invertible fractional ideals

→ $rk\oplus det: K_0(R) \xrightarrow{\sim} \mathbb{Z}\oplus Cl(R)$

$$[M] \longrightarrow \left(rkM,\ \Lambda_R^{rkM} M\right)$$

is an isomorphism of rings, where the product on $\mathbb{Z}\oplus Cl(R)$ is $(m,\mathcal{L}_1)\cdot(n,\mathcal{L}_2) = (mn, \mathcal{L}_1^n \mathcal{L}_2^m)$

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Bass defined $K_1(R) = GL(R)^{ab}$ where $GL(R) = \underrightarrow{\mathrm{colim}}\, GL_n(R)$ under the inclusions $A \longmapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$. The split exact sequence

$$0 \longrightarrow SL(R) \longrightarrow GL(R) \overset{\det}{\rightleftarrows} R^\times = GL_1(R) \longrightarrow 0$$

induces an natural isomorphism

$$\boxed{K_1(R) \cong R^\times \oplus SL(R)^{ab}}$$

$\rightsquigarrow$ For R local or $R = \mathcal{O}_K$ for K number field, $SL(R)^{ab} = 0$

$\rightsquigarrow$ For R Dedekind with fraction field F, there exists an exact sequence

$$\bigoplus_{p \in \mathrm{Max} R} K_1(R/p) \longrightarrow K_1(R) \longrightarrow K_1(F) \xrightarrow{\partial} \bigoplus_{p \in \mathrm{Max} R} K_0(R/p) \longrightarrow K_0(R) \longrightarrow K_0(F) \longrightarrow 0$$

which for $R = \mathcal{O}_K$ recovers

$$0 \longrightarrow \mathcal{O}_K^\times \longrightarrow K^\times \xrightarrow{\mathrm{div}} \bigoplus_{p} \mathbb{Z} \longrightarrow \mathbb{Z} \oplus Cl(R) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

<u>Question</u>: How can we continue on the left?

Can we generalize the result to other rings?

II. Higher K-groups

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$\boxed{1}$ The Q-construction

An <u>exact category</u> $\mathcal{C}$ is a full subcategory of an abelian category $\mathcal{A}$ containing a zero object and closed under extensions, i.e. if

$$0 \to M' \to M \to M'' \to 0 \qquad (*)$$

is exact in $\mathcal{A}$ and $M', M'' \in \mathcal{C}$, there exists $\tilde{M} \in \mathcal{C}$ isomorphic to M.

We say that a sequence $M' \rightarrowtail M \twoheadrightarrow M''$ is exact in $\mathcal{C}$, when $(*)$ is exact in $\mathcal{A}$.
An additive functor $F : \mathcal{C} \to \mathcal{D}$ is called <u>exact</u> when

$$F(M') \rightarrowtail F(M) \twoheadrightarrow F(M'')$$

is exact for every exact sequence $M' \rightarrowtail M \twoheadrightarrow M''$ in $\mathcal{C}$.

Let $\mathcal{C}$ a small exact category. Quillen constructed a new category $Q\mathcal{C}$ with the same objects as $\mathcal{C}$ and morphisms equivalence classes of diagrams $A \twoheadleftarrow D \rightarrowtail B$, where $(A \twoheadleftarrow D \rightarrowtail B) \sim (A \twoheadleftarrow D' \rightarrowtail B)$ if there exists an isomorphism $D \xrightarrow{\sim} D$ making

$$\begin{tikzcd}
 & D \arrow[dl, two heads] \arrow[dr, tail] \arrow[d, "\wr"] & \\
A & D' \arrow[l, two heads] \arrow[r, tail] & B
\end{tikzcd}$$

commute (D, D' represent the same subobject in B.) Composition is given by pullback

$$\begin{tikzcd}
 & & D \times_B D' \arrow[dl] \arrow[dr] & & \\
 & D \arrow[dl] \arrow[dr] & & D' \arrow[dl] \arrow[dr] & \\
A & & B & & C
\end{tikzcd}$$

Exact functors $\mathcal{C} \to \mathcal{D}$ induce functors $Q\mathcal{C} \to Q\mathcal{D}$.

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$\boxed{2}$ The classifying space

The **nerve** $\mathcal{NC}$ of a small category $\mathcal{C}$ is the simplicial set with

$$\mathcal{NC}_0 = \mathrm{ob}\,\mathcal{C}$$
$$\mathcal{NC}_1 = \mathrm{Mor}\,\mathcal{C}$$
$$\mathcal{NC}_n = \{(A_0 \xrightarrow{f_1} A_1 \to \cdots \xrightarrow{f_n} A_n)\}$$

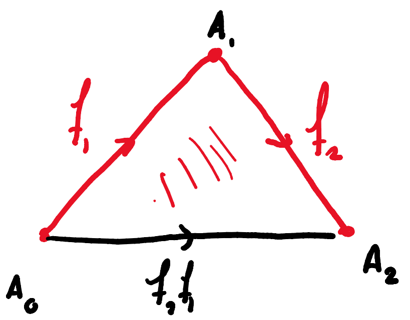
and face and degeneracy maps given by

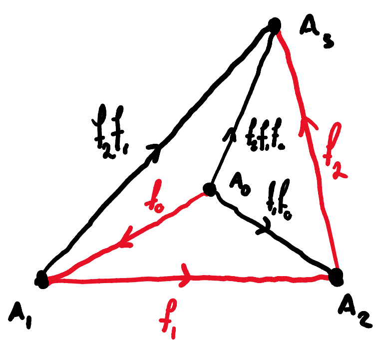
$$d_i(A_0 \to \cdots \to A_n) = (A_0 \to \cdots A_{i-1} \to A_{i+1} \to \cdots \to A_n)$$
$$s_j(A_0 \to \cdots \to A_n) = (A_0 \to \cdots \to A_i \xrightarrow{id} A_i \to \cdots \to A_n)$$

The **classifying space** $B\mathcal{C}$ is the geometric realization of $\mathcal{NC}$

$$B\mathcal{C} = \coprod_n \mathcal{NC}_n \times \Delta^n \Big/ \sim$$

where $\Delta^n = \{(x_0,\dots,x_n) \in \mathbb{R}^n : \sum x_i = 1\}$, $(d_i x, p) \sim (x, d_i p)$ and $(s_i x, p) \sim (x, s_i p)$.





The assignment $\mathcal{C} \longmapsto B\mathcal{C}$ extends to a (2-) functor sending

categories $\longrightarrow$ CW

functors $\longrightarrow$ cellular maps

nat. transformations $\longrightarrow$ homotopies

Therefore also

adjunctions $\longrightarrow$ homotopy equivalences

categories with initial obj. $\longrightarrow$ contractible spaces

3 $K(\mathcal{C})$ and $K(R)$

Let $\mathcal{C}$ a small exact category and $\mathbb{1} = \{\overset{\mathrm{id}}{\circlearrowleft}\!\!\cdot\} \to \mathcal{C}$ the functor picking out the zero object. It induces a cellular map

$$BQ\mathbb{1} = \{pt\} \longrightarrow BQ\mathcal{C}$$

making $BQ\mathcal{C}$ a pointed space.

Proposition: $BQ\mathcal{C}$ is path-connected and

$$\pi_1 BQ\mathcal{C} = \frac{\text{free abelian group on } \mathrm{Ob}\,\mathcal{C}}{\langle [B]-[A]-[C] \;:\; A \rightarrowtail B \twoheadrightarrow C \text{ exact} \rangle}$$

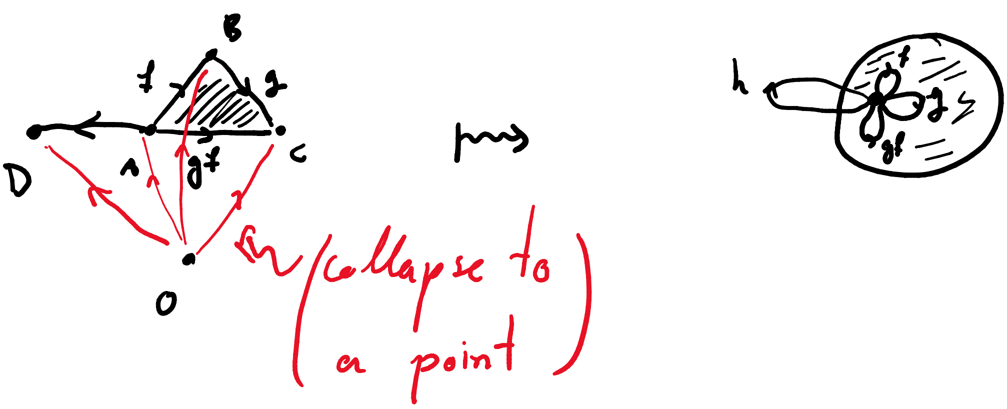
Proof: We will write $(A \rightarrowtail B)$ and $(B \twoheadleftarrow C)$ for $(A \overset{id}{\twoheadleftarrow} A \rightarrowtail B)$ and $(B \twoheadleftarrow C \overset{id}{\rightarrowtail} C)$ respectively. Every morphism in $Q\mathcal{C}$ may be factored as

$$(A \twoheadleftarrow C \rightarrowtail B) = (C \rightarrowtail B) \circ (A \twoheadleftarrow C) \quad \text{and}$$

$$(B \rightarrowtail C) \circ (A \rightarrowtail B) = (A \rightarrowtail B \rightarrowtail C)$$

$$(B \twoheadleftarrow C) \circ (A \twoheadleftarrow B) = (A \twoheadleftarrow B \twoheadleftarrow C)$$

For every $A \in \mathcal{C}$, there exists a morphism $(0 \rightarrowtail A)$, so $BQ\mathcal{C}$ is connected. Moreover those maps form a maximal tree in the 1-skeleton of $BQ\mathcal{C}$, so $\pi_1(BQ\mathcal{C})$ is generated by $f \in \operatorname{Mor} Q\mathcal{C}$ under the relations $[0 \rightarrowtail A] = 1$ and $[gf] = [f] \cdot [g]$



Reductions: $[0 \rightarrowtail A] \cdot [A \rightarrowtail B] = [0 \rightarrowtail B]$ so $[A \rightarrowtail B] = 1$

$$[0 \twoheadleftarrow A] \cdot [A \twoheadleftarrow B] = [0 \twoheadleftarrow B]$$

$$[A \twoheadleftarrow C \rightarrowtail B] = [A \twoheadleftarrow C] = [0 \twoheadleftarrow A]^{-1} \cdot [0 \twoheadleftarrow C]$$

Therefore, $\pi_1(BQ\mathcal{C})$ is generated by $[0 \twoheadleftarrow A]$ for $A \in \operatorname{Ob} \mathcal{C}$.

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$\rightsquigarrow$ If $A \rightarrowtail B \overset{f}{\twoheadrightarrow} C$ is exact, then $(C \twoheadleftarrow B)\circ(0 \rightarrowtail C) = (A \rightarrowtail B)\circ(0 \twoheadleftarrow A)$ by the diagram below

$$\begin{tikzcd}
 & & A_{= \ker f} \arrow[dl, dashed] \arrow[dr, dashed] \arrow[dd, phantom, "\lrcorner" rotate=-45] & & \\
 & 0 \arrow[dl, two heads] \arrow[dr, tail] & & B \arrow[dl, two heads, "f"] \arrow[dr, "id"] & \\
0 & & C & & B
\end{tikzcd}$$

So $[0 \twoheadleftarrow B] = [0 \twoheadleftarrow C][0 \twoheadleftarrow A]$ $\quad (*)$

In particular, for every $A, B \in \operatorname{ob}\mathcal{C}$,

$$[0 \twoheadleftarrow B][0 \twoheadleftarrow A] = [0 \twoheadleftarrow A][0 \twoheadleftarrow B]$$

as $A \rightarrowtail A \oplus B \twoheadrightarrow B$ and $B \rightarrowtail A \oplus B \twoheadrightarrow A$ are exact.

$\rightsquigarrow$ It suffices to prove no other relations exist. Given $f = (A \twoheadleftarrow D_1 \rightarrowtail B)$ and $g = (B \twoheadleftarrow D_2 \rightarrowtail C)$, let $K = \ker(D_2 \twoheadrightarrow B)$. The diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, tail, "h"] & D_2 \arrow[r, two heads] & B \arrow[r] & 0 \\
0 \arrow[r] & K \arrow[u, equal] \arrow[r, tail, "{(0,h)}"'] & D_1 \times_B D_2 \arrow[u] \arrow[r, two heads] \arrow[ur, phantom, "\urcorner"] & D_1 \arrow[u, two heads] \arrow[r] & 0
\end{tikzcd}$$

(This diagram also shows that $D_1 \times_B D_2 \in \mathcal{C}$, so composition in $Q\mathcal{C}$ is well-defined)

has exact rows so by $(*)$

$$[0 \twoheadleftarrow D_2] = [0 \twoheadleftarrow B] \cdot [0 \twoheadleftarrow K],$$
$$[0 \twoheadleftarrow D_1 \times_B D_2] = [0 \twoheadleftarrow D_1] \cdot [0 \twoheadleftarrow K].$$

So

$$[f] \cdot [g] = [gf] \iff$$

$$[0 \twoheadleftarrow A]^{-1} [0 \twoheadleftarrow D_1][0 \twoheadleftarrow B]^{-1} \cdot [0 \twoheadleftarrow D_2] = [0 \twoheadleftarrow A]^{-1} [0 \twoheadleftarrow D_1 \times_B D_2] \iff$$

$$[0 \twoheadleftarrow D_1][0 \twoheadleftarrow B]^{-1}[0 \twoheadleftarrow B][0 \twoheadleftarrow K] = [0 \twoheadleftarrow D_1][0 \twoheadleftarrow K].$$

Therefore, $[f] \cdot [g] = [gf]$ follows from $(*)$. We conclude that $\pi_1(BQ\mathcal{C})$ is generated by $[0 \twoheadleftarrow A]$ for $A \in \operatorname{Ob}\mathcal{C}$ with only relation $[0 \twoheadleftarrow B] = [0 \twoheadleftarrow C] \cdot [0 \twoheadleftarrow A]$ for $A \rightarrowtail B \twoheadrightarrow C$ exact and that the generators commute.

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<u>Remark</u>: The map $\mathrm{ob}\,\mathcal{C} \to \pi_1 BQ\mathcal{C}$ sends A to the loop

$$(0 \twoheadleftarrow A \xrightarrow{id} A) \cdot (0 \twoheadleftarrow 0 \rightarrowtail A)^{-1}$$

<u>Def</u>: The K-theory space of a small exact category $\mathcal{C}$ is the pointed CW complex

$$K(\mathcal{C}) = \Omega BQ\mathcal{C} := \mathrm{Top}_*(S^1, BQ\mathcal{C})$$

The K-theory groups are the abelian groups

$$K_n(\mathcal{C}) = \pi_n \Omega BQ\mathcal{C} = \pi_{n+1} BQ\mathcal{C}.$$

Let R commutative ring. The category $\mathcal{P}(R)$ of f.g. projective R-modules is exact, being a full subcategory of the abelian category $\mathrm{Mod}(R)$ of R-modules. We define $K(R) = K(\mathcal{C})$ for any small category $\mathcal{C}$ equivalent to $\mathcal{P}(R)$. Different choices of $\mathcal{C}$ give homotopy equivalent spaces, so

$$K_n(R) = \pi_n K(R)$$

are well-defined up to natural isomorphism.

A ring homomorphism $f: R \to S$ induces an exact functor $\mathcal{P}(R) \to \mathcal{P}(S)$ by $M \mapsto M \otimes_R S$ and therefore a cellular map

$$f^* : K(R) \to K(S)$$

well defined up to homotopy, making K into a functor

$$K : \mathrm{Rings} \to \mathrm{Ho}(\mathrm{Top})$$

III. The Main Theorems

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- <u>Additivity</u>: Let $F' \to F \to F''$ a sequence of exact functors $\mathcal{C} \to \mathcal{D}$, such that $F'(A) \rightarrowtail F(A) \twoheadrightarrow F''(A)$ is exact for every $A \in \mathcal{C}$. Then
$$F_* = F'_* + F''_* : K_n\mathcal{C} \to K_n\mathcal{D}$$
for every $n \in \mathbb{N}$.

- <u>Resolution</u>: Let $\mathcal{P} \subset \mathcal{C}$ exact subcategory, closed under extentions and kernels of $\twoheadrightarrow$, such that every $M \in \mathcal{C}$ has a finite $\mathcal{P}$-resolution, i.e. there exists an exact sequence
$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$
with each $P_i \in \mathcal{P}$. Then $K(\mathcal{P}) \simeq K(\mathcal{C})$, so $K_n(\mathcal{P}) \simeq K_n(\mathcal{C})$ for all n.

<u>Corollary</u>: Let $M(R)$ the category of finitely generated R-modules. If R is a regular noetherian ring, then
$$K(R) \xrightarrow{\sim} K(M(R)) =: G(R)$$

- <u>Devissage</u>: Let $\mathcal{B}$ a full subcategory of an abelian category $\mathcal{A}$, closed under subobjects, quotients and finite products in $\mathcal{A}$. If every object $M \in \mathcal{A}$ has a finite filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ with $M_i/M_{i-1} \in \mathcal{B}$ for all i, then $K(\mathcal{B}) \simeq K(\mathcal{A})$.

- <u>Localization</u>: Let $\mathcal{B}$ full subcategory of an abelian category $\mathcal{A}$, closed under subobjects, quotients and extentions. Then there exists a long exact sequence
$$\cdots K_{n+1}(\mathcal{A}/\mathcal{B}) \to K_n(\mathcal{B}) \to K_n(\mathcal{A}) \to K_n(\mathcal{A}/\mathcal{B}) \to \cdots$$
where $\mathcal{A}/\mathcal{B}$ is obtained from $\mathcal{A}$ by inverting morphisms whose kernel and cokernel are in $\mathcal{B}$.

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Proposition: Let R Dedekind domain. There exists a long exact sequence
$$\cdots \to K_{n+1}(F) \to \bigoplus_{p \in \mathrm{Max}\, R} K_n(R/p) \to K_n(R) \to K_n(F) \to \cdots$$

Proof: Let $\mathcal{A} = \mathcal{M}(R)$, $\mathcal{B}$ = subcategory of torsion modules

- $\mathcal{A}/\mathcal{B} \cong \mathcal{M}(F) = \mathcal{P}(F)$
- $K(R) \simeq K(\mathcal{A})$ as Dedekind rings are Noetherian regular
- Every module $M \in \mathcal{B}$ can be decomposed uniquely, as
$$M = R/p_1^{e_1} \oplus R/p_2^{e_2} \oplus \cdots \oplus R/p_n^{e_n}$$
for p_i nonzero primes of R.
- For every finite set of primes S of R, let $\mathcal{B}_S \subset \mathcal{B}$ the full subcategory on the modules whose summands correspond to primes in S. Since $\mathcal{B}_S$ is closed under quotients, subobjects and extentions, it is abelian and $\mathcal{B}_S \hookrightarrow \mathcal{B}$ is exact. Moreover
$$\mathcal{B} = \varinjlim_S \mathcal{B}_S$$
and the collection of such S is filtered, so
$$K_n(\mathcal{B}) = \varinjlim_S K_n(\mathcal{B}_S)$$

- Let $S = \{p_1, \ldots, p_m\}$, then every $M \in \mathcal{B}_S$ is of the form
$$M = R/p_1^{e_{11}} \oplus \cdots \oplus R/p_1^{e_{1r_1}} \oplus \cdots \oplus R/p_m^{e_{m r_m}}$$
so it has a filtration where the quotients are $R/p_1, \ldots, R/p_m$, obtaining by using filtrations for each term $R/p_i^{e_{ij}}$, such as
$$0 \subset p_i^{e_{ij}-1}/p_i^{e_{ij}} \subset p_i^{e_{ij}-2}/p_i^{e_{ij}} \subset \cdots \subset R/p_i^{e_{ij}}.$$

- Let $\mathcal{B}_S'$ the full subcategory of $\mathcal{B}_S$ on objects of the form $(R/p_1)^{e_1} \oplus \cdots \oplus (R/p_m)^{e_m}$. This is closed under subobjects, quotients and products, so by Devissage,

$$K_n(\mathcal{B}_S) \simeq K_n(\mathcal{B}_S')$$

- $\mathcal{B}_S' = \prod_{i=1}^m \mathcal{P}(R/p_i)$, since $\mathrm{Hom}_R\left((R/p_i)^k, (R/p_j)^l\right) = 0$ for $i \neq j$.

Therefore,
$$K(\mathcal{B}_S') = \prod_i K(R/p_i)$$
$$K_n(\mathcal{B}_S') = \bigoplus_i K_n(R/p_i)$$

- $$K_n(\mathcal{B}) = \varinjlim_{\{p_1, \dots, p_n\}} \bigoplus_i K_n(R/p_i) = \bigoplus_p K_n(R/p).$$

Remark: The morphism $K_n(R/m) \to K_n(R)$ is the transfer morphism defined as follows: If $f : R \to S$ is finite, it induces an exact functor $M(S) \to M(R)$ by restricting scalars. This induces a map
$$f_* : G(R) \to G(S)$$
and thus a map
$$f_* : K_n(R) \to K_n(S)$$
when R, S regular.

Remark: There is a generalization of this result to higher dimensional rings: If R is a regular Noetherian ring of finite dimension, then there exists a 4th quadrant spectral sequence with
$$E_2^{pq} = \bigoplus_{\mathrm{codim}(p) = p} K_{-p-q}(\mathrm{Frac}(R/p)) \Longrightarrow K_{-p-q}(R)$$

IV. K-groups of Rings of Integers

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Quillen computed

$$K_{2n-1}(\mathbb{F}_q) = \mathbb{Z}/(q^n-1)$$
$$K_{2n}(\mathbb{F}_q) = 0$$

for $n \geq 1$.

Therefore, for $F/\mathbb{Q}$ a number field and R Dedekind with $\operatorname{Frac} R = F$

(i) $0 \to K_1(R) \to K_1(F) \to \bigoplus_p K_0(R/p) \to K_0(R) \to K_0(F) \to 0$

(ii) $0 \to K_2(R) \to K_2(F) \to \bigoplus K_1(R/p) \to 0$

(iii) $0 \to K_{2n}(\mathcal{O}_F) \to K_{2n}(F) \to \bigoplus K_{2n-1}(\mathcal{O}/p) \to K_{2n-1}(\mathcal{O}_F) \to K_{2n-1}(F) \to 0$ for $n \geq 2$

<u>Remarks</u>: (1) To get that the map $\bigoplus_p K_1(\mathcal{O}_F/p) \to K_1(\mathcal{O}_F)$ is trivial and thus (i), (ii) are exact, we use that $SL(\mathcal{O}_F)^{ab} = 0$ and so $K_1(\mathcal{O}_F) = \mathcal{O}_F^\times \to F^\times = K_1(F)$ is injective

(2) Soulé proved that $K_{2n-1}(R) \to K_{2n-1}(F)$ is injective, so

- $K_n(R) \xrightarrow{\sim} K_n(F)$ for $n > 1$ odd
- $0 \to K_n(R) \to K_n(F) \to \bigoplus K_n(R/p) \to 0$ for $n > 0$ even

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What is known?

Take $R = \mathcal{O}_F$, $r = \#$ real embeddings $F \hookrightarrow \mathbb{R}$

$2s = \#$ complex embeddings $F \hookrightarrow \mathbb{C}$

• Borel-Quillen : $K_n(R)$ is finitely generated for all n and

$$\operatorname{rk} K_n(R) = \begin{cases} 1 & \text{if } n=0 \\ r+s-1 & \text{if } n=1 \\ r+s & \text{if } n \equiv 1 \pmod 4 \ \& \ n \geq 2 \\ s & \text{if } n \equiv 3 \pmod 4 \\ 0 & \text{if } n \geq 2 \text{ even} \end{cases}$$

$\mathbb{Z} \oplus \mathcal{C}\ell(R)$

$R^\times \cong \mathbb{Z}^{r+s-1} \oplus \mu(F)$

If F is totally imaginary and $n = 2i-1 \geq 3$ odd,

$$K_n(F) = \mathbb{Z}^s \oplus \mathbb{Z}/w_i(F)$$

while if F has a real embedding,

$$K_n(F) = \begin{cases} \mathbb{Z}^{r+s} \oplus \mathbb{Z}/w_i(F) & n \equiv 1 \pmod 8 \\ \mathbb{Z}^s \oplus \mathbb{Z}/2w_i(F) \oplus (\mathbb{Z}/2)^{r-1} & n \equiv 3 \pmod 8 \\ \mathbb{Z}^{r+s} \oplus \mathbb{Z}/\tfrac{1}{2}w_i(F) & n \equiv 5 \pmod 8 \\ \mathbb{Z}^s \oplus \mathbb{Z}/w_i(F) & n \equiv 7 \pmod 8 \end{cases}$$

for

$$w_i(F) = \prod w_i^{(\ell)}(F)$$

where

$$w_i^{(\ell)}(F) = \sup\{\ell^\nu \mid Gal(F(\zeta_{\ell^\nu})/F) \text{ has exponent dividing } i\}$$

References

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References

- ⇝ Grayson : Quillen's work on alg. K-theory
- ⇝ Weibel : The K-book
- ⇝ Wabel : Alg. K-theory of rings of integers in local & global fields

Epilogue - Fun / Useful Fact

For X regular scheme of finite type, the assignment

$$U \longrightarrow K_n(U)$$

forms a presheaf of groups. If $\mathcal{K}_n$ denotes its sheafification, then

$$H^n(X; \mathcal{K}_n) \simeq CH_n(X)$$

is the Chow group of cycles of codimension n up to rational equivalence.