808F 20 FALL ASSIGNMENT 2

In these exercises we show some results on temperedness and square-integrability for irreducible unitary representations of $\mathrm{GL}_2(\mathbb{R})^+$.

1. Principal series

Let $G = \mathrm{GL}_2(\mathbb{R})^+$ and $K = \{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \theta \in [0, 2\pi)\} \cong \mathrm{SO}(2)$.

1.1. **Matrix identity.** Suppose that $x, \tilde{x} \in \mathbb{R}$, $a_1, a_2, \tilde{a_1}, \tilde{a_2} \in \mathbb{R}_{>0}$, $\theta, \phi \in [0, 2\pi)$ satisfy

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} = \begin{bmatrix} 1 & \tilde{x} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{a}_1 & 0 \\ 0 & \tilde{a}_2 \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \tag{1.1}$$

Verify the following:

$$\tilde{a}_2 = \sqrt{a_2^2(x\sin\theta - \cos\theta)^2 + a_1^2\sin^2\theta}, \quad \tilde{a}_1 = \frac{a_1 a_2}{\tilde{a}_2}, \quad \tan\phi = \frac{a_1\tan\theta}{a_2(1 - x\tan\theta)}$$

Deduce

$$\tilde{a}_1 = \sqrt{(a_1\cos\phi + a_2 x\sin\phi)^2 + a_2^2\sin^2\phi}, \quad d\theta = \frac{a_1 a_2}{\tilde{a}_1^2}d\phi = \frac{a_1 a_2}{(a_1\cos\phi + a_2 x\sin\phi)^2 + a_2^2\sin^2\phi}d\phi$$

where in the last equality, we view x, a_1, a_2 as constants and ϕ as variable.

1.2. **Integral identity.** For $j = 1, 2$, let $\chi_j : \mathbb{R}^\times \to \mathbb{C}^\times$ be the quasi-character defined by $\chi_j(x) = |x|^{s_j}\mathrm{sgn}(x)^{\epsilon_j}$ where $s_j \in \mathbb{C}$ and $\epsilon_j \in \{0, 1\}$. Consider the normalized induction $H(s_1, s_2, \epsilon_1, \epsilon_2)^\infty := i_B^G(\chi_1 \boxtimes \chi_2)$, which consists of functions $f \in C^\infty(G)$ such that

$$f\left(\begin{bmatrix} a_1 & b \\ 0 & a_2 \end{bmatrix} g\right) = |a_1|^{s_1 + \frac{1}{2}}|a_2|^{s_2 - \frac{1}{2}}\mathrm{sgn}(a_1)^{\epsilon_1}\mathrm{sgn}(a_2)^{\epsilon_2}f(g), \quad \forall g \in G$$

Let

$$g = u\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}, \quad u, y \in \mathbb{R}_{>0}, x \in \mathbb{R}, \alpha \in [0, 2\pi).$$

Denote $\mu := s_1 + s_2$ and $s := \frac{1}{2}(s_1 - s_2 + 1)$. Show that for all $f_1, f_2 \in H(s_1, s_2, \epsilon_1, \epsilon_2)^\infty$,

$$f_1(k_\theta g) = u^\mu y^{-s}[(y\cos\phi + x\sin\phi)^2 + \sin^2\phi]^s f_1(k_\phi), \quad \theta, \phi \text{ as in (1.1).}$$

$$\int_0^{2\pi} f_1(k_\theta g)\overline{f_2(k_\theta g)}\frac{d\theta}{2\pi} = u^{2\mathrm{Re}(\mu)}\int_0^{2\pi}[y^{-1}(y\cos\phi + x\sin\phi)^2 + y^{-1}\sin^2\phi]^{2\mathrm{Re}(s)-1}f_1(k_\phi)\overline{f_2(k_\phi)}\frac{d\phi}{2\pi}$$

where $k_\theta := \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

1.3. **Unitary principal series.** Consider the inner product $(f_1, f_2) := \int_K f_1(k)\overline{f_2(k)}dk$ on $H(s_1, s_2, \epsilon_1, \epsilon_2)^\infty$. Conclude that

(1) Any $g \in G$ induces a bounded linear operator on $H(s_1, s_2, \epsilon_1, \epsilon_2)^\infty$.
(2) When $s_1, s_2 \in i\mathbb{R}$, the inner product is G-invariant.

1.4. **Asymptotics of matrix coefficients.** Consider the pairing

$$\langle \_, \_ \rangle : H(s_1, s_2, \epsilon_1, \epsilon_2)^\infty \times H(-s_1, -s_2, \epsilon_1, \epsilon_2)^\infty \to \mathbb{C}$$

Defined by $\langle f_1, f_2 \rangle := \int_0^{2\pi} f_1(k_\theta) f_2(k_\theta) \frac{d\theta}{2\pi}$. Conclude that this pairing is non-degenerate G-equivariant. Hence realizes the duality between the two principal series.

Suppose moreover that f_1, f_2 are eigen-vectors for K, consider the corresponding matrix coefficient for $H(s_1, s_2, \epsilon_1, \epsilon_2)^\infty$ defined as

$$\varphi(g) := \langle g \cdot f_1, f_2 \rangle, \quad \forall g \in G$$

Denote $a_y := \begin{bmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{bmatrix}$. Let $\sigma := \mathrm{Re}(s)$ and assume $0 < \sigma < 1$. Show that when $y \geq 1$,

$$\begin{aligned} |\varphi(a_y)| &\leq \frac{2}{\pi} ||f_1||_\infty ||f_2||_\infty y^{1-\sigma} \int_0^{\frac{\pi}{2}} (y^2 \sin^2\theta + \cos^2\theta)^{\sigma-1} d\theta \\ &\leq \begin{cases} \frac{2}{\pi} ||f_1||_\infty ||f_2||_\infty [y^{-\frac{1}{2}}(1-y^{-2})^{-\frac{1}{2}} + \frac{\pi}{2} y^{-\frac{1}{2}}(\log y + \log \frac{\pi}{2})] & \text{if } \sigma = \frac{1}{2} \\ \frac{2}{\pi} ||f_1||_\infty ||f_2||_\infty [y^{-\sigma}(1-y^{-2})^{\sigma-1} + \frac{y^{\sigma-1}}{2\sigma-1}(\frac{\pi}{2})^{2\sigma-2}((\frac{\pi}{2})^{2\sigma-1} - y^{1-2\sigma})] & \text{if } \sigma \neq \frac{1}{2} \end{cases} \end{aligned}$$

(Hint: For the second inequality, break the integral $\int_0^{\frac{\pi}{2}}$ into $\int_0^{y^{-1}}$ and $\int_{y^{-1}}^{\frac{\pi}{2}}$, estimate the first part using Taylor expansion of $\cos\theta$ and the second part using the inequality $\sin\theta \geq \frac{2\theta}{\pi}$ for $0 < \theta \leq \frac{\pi}{2}$)

Conclude that as $y \to +\infty$, the asymptotic order of $\phi(a_y)$ is given by

$$\phi(a_y) \sim \begin{cases} \frac{\log y}{\sqrt{y}}, & \text{if } \sigma = \frac{1}{2} \\ y^{-\sigma}, & \text{if } 0 < \sigma < \frac{1}{2}, \\ y^{\sigma-1}, & \text{if } \frac{1}{2} < \sigma < 1 \end{cases}$$

1.5. **Temperedness.** Recall the Cartan decomposition $\mathrm{SL}_2(\mathbb{R}) = KA_+K$ where $A_+ := \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, a \geq 1 \right\}$. From this one gets the following integral formula

$$\int_{\mathrm{SL}_2(\mathbb{R})} \varphi(g) dg = \int_K \int_1^\infty \int_K \varphi(k_1 a_y k_2)(y - y^{-1}) dk_1 \frac{dy}{y} dk_2$$

where $a_y := \begin{bmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{bmatrix}$.

Use this formula to show that the unitary principal series and limit of discrete series (i.e. when $\mathrm{Re}(s) = \frac{1}{2}$) are tempered, i.e. any K-finite matrix coefficients belong to $L^{2+\epsilon}(\mathrm{SL}_2(\mathbb{R}))$ for all $\epsilon > 0$. Also show that the complementary series (i.e. when $s \in (0,1), s \neq \frac{1}{2}$) are not tempered.

2. Discrete series

For each integer $n \geq 2$, we consider the realization of the discrete series $D^\pm(n)$ for $G := \mathrm{GL}_2(\mathbb{R})$ as certain space of holomorphic functions on the upper half plane $\mathcal{H}$. For simplicity, we require the positive scalar matrices to act trivially, i.e. $\mu = 0$ in standard notation.

Recall that $L^2_{\mathrm{hol}}(\mathcal{H}, n)$ is the space of holomorphic functions on $\mathcal{H}$ that are square integrable with respect to the measure $\mu_n := y^{n-2} dx dy$. In particular, μ_n induces an inner product on $L^2_{\mathrm{hol}}(\mathcal{H}, n)$. Define the representation π_n of G on $L^2_{\mathrm{hol}}(\mathcal{H}, n)$ by

$$(\pi_n(g)f)(z) := (f|_n {}^t g)(z) = (ad - bc)^{\frac{n}{2}} (bz + d)^{-n} f(\frac{az + c}{bz + d}), \quad \forall g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$$

2.1. Use Cauchy integral formula to show that $L^2_{\mathrm{hol}}(\mathcal{H}, n)$ is complete, so that it is a Hilbert space.

2.2. Show that the Caley transform $z \mapsto w = \frac{z-i}{z+i} = u + iv$ induces an isomorphism of Hilbert spaces from $L^2_{\mathrm{hol}}(\mathcal{H}, n)$ to the Hilbert space $L^2_{\mathrm{hol}}(\mathbb{D}, n)$ consisting of holomorphic functions on the open unit disc $\mathbb{D}$ that are square integrable with respect to the measure $d\nu_n := \frac{4(1-|w|^2)^{n-2} du dv}{|1-w|^{2n}}$.

2.3. Check that the Caley transform $\mathcal{C} := \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$ induces

$$\mathcal{C} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \mathcal{C}^{-1} = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

Also check that (conjugation inside $\mathrm{GL}_2(\mathbb{C})$)

$$\mathcal{C} G \mathcal{C}^{-1} = U(1,1)^+ := \left\{ \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix}, \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 > 0 \right\}$$

2.4. Using the isomorphism in § 2.2, we get a representation ρ_n of G on $L^2_{\mathrm{hol}}(\mathbb{D}, n)$. Let $\varphi \in L^2_{\mathrm{hol}}(\mathbb{D}, n)$ and let $f(z) := \varphi(\mathcal{C} z)$. Show that for all $w \in \mathbb{D}$ we have

$$(\rho_n(g)\varphi)(w) = (\pi_n(g) f)(\mathcal{C}^{-1} w) = (\det g)^{\frac{n}{2}} \frac{(1-w)^n}{((bi-d)w + bi + d)^n} \varphi(\mathcal{C}\, {}^t g \mathcal{C}^{-1} w), \quad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$$

Deduce that for all $w \in \mathbb{D}$,

$$(\rho_n(k_\theta)\varphi)(w) = \frac{(1-w)^n}{(e^{i\theta} - e^{-i\theta} w)^n} \varphi(e^{-2i\theta} w), \quad k_\theta := \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

2.5. Show that the functions $\{w^m (1-w)^n\}_{m=0}^{\infty}$ form an orthonormal basis for $L^2_{\mathrm{hol}}(\mathbb{D}, n)$ and $w^m(1-w)^n$ has weight $-2m - n$ under the action of $K = \mathrm{SO}(2)$. Deduce that the functions $\{(\frac{z-i}{z+i})^m \frac{(2i)^n}{(z+i)^n}\}_{m=0}^{\infty}$ form an orthonormal basis of $L^2_{\mathrm{hol}}(\mathcal{H}, n)$ consisting of eigenvectors for K.

Conclude that $\pi_n \cong D^-(n)$.

2.6. Show that the map $f \mapsto \varphi_f$ defines a G-equivariant embedding of Hilbert spaces $L^2_{\mathrm{hol}}(\mathcal{H}, n) \hookrightarrow L^2(G/Z_G^+)$ (maybe up to a scalar) where G acts on the target by right regular representation and

$$\varphi_f(g) := f({}^t g i) \det(g)^{\frac{n}{2}} (bi + d)^{-n}, \quad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$$

Conclude that $D^-(n)$ is isomorphic to a sub-representation of $L^2(G/Z_G^+)$ and hence square-integrable. Twist the representation π_n by $\eta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, draw similar conclusion for $D^+(n)$.