808F 20 FALL ASSIGNMENT 1

1. Fourier coefficients of Poincaré series

Let $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ and $\Gamma_\infty := \pm \begin{bmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{bmatrix} \subset \Gamma$. Let $k > 2$ be an even integer and $m \in \mathbb{Z}_{\geq 0}$. Recall that the Poincaré series (of weight k) on the upper half plane $\mathcal{H}$ is defined by

$$P_m(z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} j(\gamma, z)^{-k} e^{2\pi i m \gamma z}.$$

As special cases, we have the (normalized) Eisenstein series

$$P_0(z) = G_k(z) = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z}^2 \\ (c,d)=1}} (cz+d)^{-k}, \quad E_k(z) = \zeta(k) G_k(z) = \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z}^2 \\ (m,n) \neq (0,0)}} (mz+n)^{-k}$$

In this problem we compute the Fourier coefficients of $P_m(z)$.

1.1. Split the sum according to the double coset $\Gamma_\infty \backslash \Gamma / \Gamma_\infty$ and show that

$$P_m(z) = e^{2\pi i m z} + \sum_{c=1}^{\infty} \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} \sum_{u \in \mathbb{Z}} (cz+d+uc)^{-k} \exp[2\pi i m(\frac{\bar{d}}{c} - \frac{1}{c(cz+d+uc)})]$$

Here the second summand is over a set of representatives of $(\mathbb{Z}/c\mathbb{Z})^\times$ in the interval $[0,c)$ (only one term $d=0$ when $c=1$) and $\bar{d}$ denotes any integer such that $\bar{d}d \equiv 1 \mod c$.

1.2. Let $p(m,n)$ be the n-th Fourier coefficient of $P_m(z)$ so that $P_m(z) = \sum_{n \in \mathbb{Z}} p(m,n) e^{2\pi i n z}$. Show that

$$p(m,n) = \delta_{m,n} + \sum_{c=1}^{\infty} S(m,n;c) J_c(m,n)$$

where

$$J_c(m,n) := \int_{-\infty+iy}^{\infty+iy} (cv)^{-k} \exp(\frac{-m}{c^2 v} - nv) dv$$

is independent of $y > 0$ and

$$S(m,n;c) := \sum_{\substack{a,d \in \mathbb{Z}/c\mathbb{Z} \\ ad \equiv 1 \mod c}} \exp[2\pi i (\frac{ma+nd}{c})]$$

is the *Kloosterman sum*.

1.3. Show that $J_c(m,n) = 0$ if $n \leq 0$. If $n > 0$, use Taylor expansion of $e^{2\pi i z}$ to calculate certain residue and show that

$$J_c(0,n) = (\frac{2\pi}{ic})^k \frac{n^{k-1}}{(k-1)!}$$

1.4. Check that when $m > 0$ and $n > 0$, then

$$J_c(m,n) = \frac{2\pi}{i^k c}(\frac{n}{m})^{\frac{k-1}{2}} J_{k-1}(\frac{4\pi\sqrt{mn}}{c})$$

where the *Bessel function of order* ν is defined by

$$J_\nu(x) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!\Gamma(\ell+1+\nu)}(\frac{x}{2})^{\nu+2\ell}.$$

1.5. Conclude that when $m > 0$, $P_m(z)$ is a cusp form and for all $n > 0$, its n-th Fourier coefficient is given by

$$p(m,n) = (\frac{n}{m})^{\frac{k-1}{2}}(\delta_{mn} + 2\pi i^{-k}\sum_{c=1}^{\infty} c^{-1}S(m,n;c)J_{k-1}(\frac{4\pi\sqrt{mn}}{c}))$$

1.6. Show that for any integer $k > 2$ (not necessarily even) and $n > 0$,

$$\zeta(k)\sum_{c=1}^{\infty} c^{-k}S(0,n;c) = n^{1-k}\sigma_{k-1}(n)$$

where $\sigma_r(n) := \sum_{d|n} d^r$. Conclude that the Fourier expansion of the Eisenstein series $G_k(z)$ is given by

$$G_k(z) = P_0(z) = 1 + \frac{(2\pi)^k(-1)^{\frac{k}{s}}}{\zeta(k)(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)e^{2\pi i n z}.$$

1.7. Recall that for any $f \in S_k(\Gamma)$ and $g \in M_k(\Gamma)$, the Petersson inner product is defined by

$$(f,g) := \int_{\Gamma\backslash\mathcal{H}} f(z)g(z)\mathrm{Im}(z)^k\frac{dxdy}{y^2}.$$

Let $f \in S_k(\Gamma)$ be a cusp form with Fourier expansion $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$. Show that $(f, P_0) = 0$ and when $m > 0$,

$$(f, P_m) = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}}a_m = \frac{(k-2)!}{(4\pi m)^{k-1}}a_m.$$

Conclude that $S_k(\Gamma)$ is spanned by $\{P_m\}_{m\geq 1}$.

2. Differential operators

Let $G := \mathrm{GL}_2(\mathbb{R})^+$ and $K := \mathrm{SO}(2)$. Recall that any element $g \in G$ can be written uniquely as

$$g = \lambda\begin{bmatrix}1 & x\\0 & 1\end{bmatrix}\begin{bmatrix}y^{\frac{1}{2}} & 0\\0 & y^{-\frac{1}{2}}\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{bmatrix} \quad \text{where } \lambda, y \in \mathbb{R}_{>0}, x \in \mathbb{R}, \theta \in [0, 2\pi). \tag{2.1}$$

Hence as a smooth manifold, we have $G \cong \mathbb{R}_{>0} \times \mathcal{H} \times K$ and we use λ, x, y, θ as coordinates on G. Moreover, we have the standard basis for $\mathfrak{g} = \mathfrak{gl}_2$:

$$Z = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \quad H = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}, \quad E = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \quad F = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$$

and an alternative basis for $\mathfrak{sl}_2(\mathbb{C})$ (where $C = \begin{bmatrix}1 & -i\\1 & i\end{bmatrix}$ denotes the Caley transform):

$$W = C^{-1}HC = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}, \quad R = C^{-1}EC = \frac{1}{2}\begin{bmatrix}1 & i\\i & -1\end{bmatrix}, \quad L = C^{-1}FC = \frac{1}{2}\begin{bmatrix}1 & -i\\-i & -1\end{bmatrix}$$

2.1. Identify $\mathfrak{g}_{\mathbb{C}}$ as left invariant differential operators on G, defined via right regular representation. Verify the following formulas:

$$W = -i\frac{\partial}{\partial\theta},$$

$$R = e^{2i\theta}(iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{2i}\frac{\partial}{\partial\theta}),$$

$$L = e^{-2i\theta}(-iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{1}{2i}\frac{\partial}{\partial\theta})$$

Recall that $\Delta := -\frac{1}{4}(W^2 + 2RL + 2LR)$. Check that

$$\Delta = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + y\frac{\partial^2}{\partial x\partial\theta}.$$

2.2. Let $f \in M_k(\Gamma)$ be a holomorphic modular form of weight k. Let $\varphi_f \in C^\infty(G)$ be the associated automophic form defined by $\varphi_f(g) = f(gi)j(g,i)^{-k}\det(g)^{\frac{k}{2}}$. Show that for g given as in (2.1), we have

$$\varphi_f(g) = \lambda^k f(x+iy)y^{\frac{k}{2}}e^{ik\theta}.$$

Deduce that

$$L\varphi_f = 0 \quad \text{and} \quad \Delta\varphi_f = \frac{k}{2}(\frac{k}{2}-1)\varphi_f$$

3. Eisenstein series of weight 0

Let $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ and $\Gamma_\infty := \pm\begin{bmatrix}1 & \mathbb{Z}\\ 0 & 1\end{bmatrix} \subset \Gamma$. Let $\varphi \in C^\infty(\mathbb{R}_{>0})$. For all $g \in G = \mathrm{GL}_2(\mathbb{R})^+$, define

$$E_\varphi(g) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \varphi(\mathrm{Im}(\gamma g i))$$

provided the sum converges absolutely.

3.1. Show that if $|\varphi(y)|$ is bounded above by (some constant multiple of) y^α as $y \to 0$ for some $\alpha > 1$, then the sum defineing E_φ converges absolutely.

3.2. Take $\varphi(y) = y^s$, $s \in \mathbb{C}$. Then $E_\varphi = E_s$ comes from the non-holomorphic Eisenstein series on $\mathcal{H}$ defined by:

$$E(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma z)^s = \frac{1}{2}\sum_{(c,d)=1} \frac{\mathrm{Im}(z)^s}{|cz+d|^{2s}}.$$

In other words, $E_s(g) = E(gi, s)$. Show that this converges absolutely when $\mathrm{Re}(s) > 1$ and in this case use the invariance property of Δ to show that

$$\Delta E_s = s(1-s)E_s.$$

3.3. More generally, take $\varphi(y) = (\log y)^n y^s$ where $n \in \mathbb{Z}_{\geq 0}$ and get

$$E_{n,s}(g) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\log(\mathrm{Im}(\gamma \mathrm{gi})))^n \mathrm{Im}(\gamma g i)^s$$

Show that this converges absolutely when $\mathrm{Re}(s) > 1$ and in this case, we have

$$(\Delta - s(1-s))^{n+1} E_{n,s} = 0$$

and when $n > 0$, no smaller power of $\Delta - s(1-s)$ annihilates $E_{n,s}$. In particular, when $n > 0$, the center of universal enveloping algebra $\mathcal{Z}_{\mathfrak{g}}$ does not act semi-simply on the $(\mathfrak{g}, K)$-module generated by $E_{n,s}$.