Homework 5

Math 607 (section 0101), Spring 2022

Due Wednesday, March 16

You are encouraged to think about problems marked with a (*), but they are not to be handed in.

1. Consider the Veronese embedding $v_d : \mathbb{P}^1 \to \mathbb{P}^d$.

- **(a)** Find the degree of the morphism v_d.
- **(b)** Show that the image $v_d(\mathbb{P}^1) \subset \mathbb{P}^1$ is a subvariety.
- **(c)** Find the degree of the curve in the image $v_d(\mathbb{P}^1) \subset \mathbb{P}^1$.

2. Consider the curve C cut out by $zy^2 = x^3$ in $\mathbb{P}^2$.Calculate the intersection including the multiplicities of C with the hyperplanes

- **(a)** $x = 0$
- **(b)** $y = 0$
- **(c)** $z = 0$

3. Let R me a commutative ring and $x \in R$ is not a zero divisor. Let M be R-module. Prove the following isomorphisms:

- **(a)** $\mathrm{Tor}_0(R/x, M) = M/xM$
- **(b)** $\mathrm{Tor}_1(R/x, M) = M[x]$
- **(c)** $\mathrm{Tor}_i(R/x, M) = 0$ for all $i \geq 2$.

4. (Hartshone I.3.14) Projection from a point. Let $\mathbb{P}^n$ be a hyperplane in $\mathbb{P}^{n+1}$ and let $P \in \mathbb{P}^{n+1} - \mathbb{P}^n$. Define a mapping $\phi : \mathbb{P}^{n+1} - \{P\} \to \mathbb{P}^n$ by $\phi(Q) =$ the intersection of the unique line containing P and Q with $\mathbb{P}^n$.

- **(a)** Show that ϕ is a morphism.
- **(b)** Let $Y \subset \mathbb{P}^3$ be the twisted cubic curve which is the image of the 3-uple veronese embedding. If t, u are the coordinates on $\mathbb{P}^1$, we say that Y is the curve given *parametically* by $(x, y, z, w) = (t^3, t^2u, tu^2, u^3)$. Let $P = (0, 0, 1, 0)$ and let $\mathbb{P}^2$ be the hyperplane $z = 0$. Show that the projection of Y from P is a cuspidal cubic curve in the place and find its equation.

5. (Hartshorne I.4.4) A variety Y is *rational* if it is birationally equivalent to $\mathbb{P}^n$ for some n (or equivalently if the function field $k(Y)$ is a pure transcendental extension of k.)

- **(a)** Any conic in $\mathbb{P}^2$ is a rational curve.
- **(b)** The cuspidal cubic $y^2 = x^3$ is a rational curve.
- **(c)** Let Y be the nodal cubic curve $y^2z = x^2(x+z)$ in $\mathbb{P}^2$. Show that the projection ϕ from the point $P = (0, 0, 1)$ to the line $z = 0$ induces a birational map from Y to $\mathbb{P}^1$. Thus Y is a rational curve.

6. (Hartshorne I.2.15) The Quadric Surface in $\mathbb{P}^3$ Consider the surface Q in $\mathbb{P}^3$ defined by the equation $xy - zw = 0$.

(a) Show that Q is equal to the Segre embedding ψ of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ for suitable choice of coordinates.

(b) Show that Q contains two families of lines (a *line* is a linear variety of dimension 1) $\{L_t\}, \{M_t\}$ each parametrized by $t \in \mathbb{P}^1$ with the properties that if $L_t \neq L_u$, then $L_t \cap L_u = \emptyset$; if $M_t \neq M_u$, then $M_t \cap M_u = \emptyset$; and for all t, u, $L_t \cap M_u =$ one point.

(c) Show that Q contains other curves besides these lines and deduce that the Zariski topology on Q is not homeomorphic via ψ to the product topology on $\mathbb{P}^1 \times \mathbb{P}^1$ (where each $\mathbb{P}^1$ has its Zariski topology.)