Homework 4

Math 607 (section 0101), Spring 2022

Due Wednesday, March 9

You are encouraged to think about problems marked with a (*), but they are not to be handed in.

1. The point of this problem is to explain how Weil divisors D whose multiples mD become Cartier for $m \in \mathbb{Z}_{>1}$ appear in the study of *quotient singularities* by giving a specific example. Assume char $k \neq 2$.

(a) Show that the scheme $X = \operatorname{Spec} k[x, y, x]/(xy - z^2)$ is normal but not regular.

(b) Consider the following automorphism $g : k[u, v] \to k[u, v]$ given by $f(u, v) \mapsto f(-u, -v)$ for a polynomial $f(u, v) \in k[u, v]$. Let $k[u, v]^g$ be the g-invariant part of $k[u, v]$. Find the subring $k[u, v]^g \subset k[u, v]$.

(c) The ring automorphism $g : k[u, v] \to k[u, v]$ induces an isomorphism ϕ of $\mathbb{A}^2 = \operatorname{Spec} k[u, v]$. Let $< \phi >$ be the cyclic group of order 2 generated by ϕ. For points x, y in $\mathbb{A}^2$ one can define an equivalence relation between x and y by either $x = y$ or $y = \phi(x)$. Let $\mathbb{A}^2/ < \phi >$ be the quotient space induced by this relation. The scheme structure is $\mathbb{A}^2/ < \phi >= \operatorname{Spec}(k[u, v]^g)$. Prove that $\mathbb{A}^2/ < \phi >$ is isomorphic to X.

2. Tautological line bundle on $\mathbb{P}^n_k$. We describe $\mathbb{A}^{n+1}_k$ with coordinates $w = (w_0, \ldots, w_n)$ and $\mathbb{P}^n_k$ with coordinates $z = [z_0 : \ldots : z_n]$. We consider the variety $X \subset \mathbb{A}^{n+1} \times \mathbb{P}^n$ defined by equations $w_i z_j - w_j z_i = 0$ for all $0 \leq i, j \leq n$. Note that this is the blow up of $\mathbb{A}^{n+1}_k$ at the origin. Using the projection $\pi : X \to \mathbb{P}^n$ given by $(w, z) \in X \mapsto z \in \mathbb{P}^n$, X gets the structure of a line bundle as in topology.

(a) Explain why set theoretically X can be described as the set of couples $[l] \times v \in \mathbb{A}^{n+1} \times \mathbb{P}^n$ where the points $[l] \in \mathbb{P}^n$ are identified with the lines l in $\mathbb{A}^{n+1}$ and the point $v \in \mathbb{A}^{n+1}$ lies on the line l. (This means that X is an example of the incidence correspondence.)

(b) Over the standard sets $U_i = D_+(w_i) \subset \mathbb{P}^n$ we have the trivialization isomorphisms $\phi_i : \pi^{-1}(U_i) \to \mathbb{A}^1 \times U_i$ given by $(w, z) \mapsto (w_i, z)$. Describe the transitions functions $\phi_i \circ \phi_j^{-1}$ on the coordinates.

(c) Using the transition functions $\phi_i \circ \phi_j^{-1}$, find the corresponding invertible sheaf on $\mathbb{P}^n$ with the "same" transition functions.

3. The point of this problem is to study the relationship between Cartier divisors, Weil divisors, and invertible sheaves on $\mathbb{P}^n$.

(a) Describe each invertible sheaf $\mathcal{O}_{\mathbb{P}^n}(m)$ as a Cartier divisor.

(b) Using the Cartier divisor description of $\mathcal{O}_{\mathbb{P}^n}(m)$, calculate the corresponding Weil divisor.

(c) Which of the invertible sheaves $\mathcal{O}_{\mathbb{P}^n}(m)$ correspond to effective Cartier divisors and why?

(d) What are all sections and all rational sections of $\mathcal{O}_{\mathbb{P}^n}(m)$?

(e) Describe the Weil divisors corresponding to the rational sections of $\mathcal{O}_{\mathbb{P}^n}(m)$.

4. Calculate the Weil class group of $X = \operatorname{Spec} k[x,y,z,w]/(xy - zw)$ using the following steps.

(a) Let Z be prime divisor associated to the ideal $p = (y, z)$. Using the exact sequence for the class groups of $Z, X, X - Z$, prove that $Cl(X)$ is a cyclic $\mathbb{Z}$-module.

(b) Now we need to show that this module is not torsion. Denote $A = k[x,y,z,w]/(xy - zw)$ and $S = k[x,y,z,w]$. Explain why it is sufficient to show that n-th symbolic power $p^{(n)} = p^n A_p \cap A$ of p is not principal for all $n \geq 1$.

(c) We will prove the goal of (b) by contradiction. Assume that the image of $f \in S$ in A generates $p^{(n)}$. Then using $(y^n, z^n) \subset (f, xy - zw) \subset (f, x, w)$ and Krull's height theorem, reach a contradiction.

5. Derivations. Suppose A is a B-algebra for rings A, B. For any A-module M, a B-derivation of A into M is defined as a B-linear map $d : A \to M$ such that

(i) (additivity) $d(a + a') = da + da'$

(ii) (Leibniz rule) $d(aa') = ada' + a'da$

(iii) (triviality) $db = 0$ for every $b \in \phi(B) \subseteq A$ for the structure homomorphism $\phi : B \to A$.

The module of Kahler differentials of A over B is an A-module $\Omega_{A/B}$ together with a B-derivation $d : A \to \Omega_{A/B}$. This module is the "representing object" of the function of B-derivations $M \mapsto Der_B(A, M)$. We can construct the module $\Omega_{A/B}$ as the free module generated by the symbols $\{da | a \in A\}$ and take the quotient by the submodule generated by the equivalence relations in $(i - iii)$ i.e. $d(a + a') - da - da',\ d(aa') - ada' - a'da,\ db$. Then the map $d : A \to \Omega_{A/B}$ which sends a to da is a B-derivations. The pair $(\Omega_{A/B}, d)$ is unique up to unique isomorphism. Prove the following isomorphisms:

(a) Let $A = B/I$, then prove that $\Omega_{A/B} = 0$.

(b) Let $A = B[x_1, \ldots, x_n]$. Then prove that $\Omega_{A/B} = Adx_1 \oplus Adx_2 \oplus \ldots \oplus Adx_n$ is a free A module of rank n.

(c) Let $B = k$ is a field of characteristic$\neq 2$ and let $A = k[x,y]/(y^2 - x^3 + x)$. Then prove that the module $\Omega_{A/B}$ is generated by dx and dy subject to the relation $2ydy = (3x^2 - 1)dx$. Prove additionally that the sheaf $\widetilde{\Omega_{A/B}}$ is an invertible sheaf on $\operatorname{Spec} A$.

(d) Let $B = k$ is a field of characteristic$\neq 2$ and let $A = k[x,y]/(y^2 - x^3)$. Then prove that the module $\Omega_{A/B} \cong (Adx \oplus Ady)/(2ydy - 3x^2dx)$. Prove additionally that the sheaf $\widetilde{\Omega_{A/B}}$ is not a locally free sheaf on $\operatorname{Spec} A$.

6. Cotangent sheaf $\Omega_{X/k}$. If X is a scheme of finite type over algebraically closed field k, then we can cover X by open affine sets $U = \operatorname{Spec} A$ where A is a finitely generated k-algebra and thus we have the finitely generated A-module $\Omega_{A/k}$ described in the previous problem. We can glue the sheaves $\widetilde{\Omega_{A/k}}$ to a coherent sheaf $\Omega_{X/k}$, using the gluing of the affine sets. In general the cotangent sheaf $\Omega_{X/k}$ is not an invertible sheaf, it is a locally free sheaf of rank $= \dim X$ for smooth X and its top wedge power called the canonical sheaf $\omega_{X/k}$ is an invertible sheaf.

(a) Assuming that the cotangent sheaf $\Omega_{\mathbb{P}^1/k}$ is an invertible sheaf, calculate the degree of the divisor of a nonzero rational section. Use that to calculate the sheaf $\Omega_{\mathbb{P}^1/k}$.

(b) Let C be a smooth plane curve i.e. a smooth curve given by $f(x,y) = 0$ in $\mathbb{A}^2_k$. Prove that the cotangent sheaf $\Omega_{C/k}$ is invertible.