Homework 3

Math 607 (section 0101), Spring 2022

Due Wednesday, March 2

You are encouraged to think about problems marked with a (*), but they are not to be handed in.

1. (Hartshorne Ex 6.10 in II.6) The Grothendieck group $K(X)$. Let X be a noetherian scheme. We define $K(X)$ to be the quotient of the free abelian group generated by all coherent sheaves on X, by the subgroup generated by all expressions $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$, whenever there is an exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of coherent sheaves on X. If $\mathcal{F}$ is coherent sheaf, we denote by $\gamma(\mathcal{F})$ its image in $K(X)$.

- **(a)** If $X = \mathbb{A}^1_k$. then $K(X) \cong \mathbb{Z}$
- **(b)** If X is any integral scheme and $\mathcal{F}$ is a coherent sheaf, we define the *rank* of $\mathcal{F}$ to be $\dim_K \mathcal{F}_\eta$ where η is the generic point of X and $K = \mathcal{O}_\eta$ is the function field of X. Show that the rank function defines a surjective homomorphism rank: $K(X) \to \mathbb{Z}$
- **(c)** (*) If Y is a closed subscheme of X, there is an exact sequence sequence

$$K(Y) \to K(X) \to K(X - Y) \to 0$$

where the first map is extension by zero, and the second map is restriction. [Hint: For exactness in the middle show that if $\mathcal{F}$ is a coherent sheaf on X, whose support is contained in Y, then there is a finite filtration $\mathcal{F} = \mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \ldots \supseteq \mathcal{F}_n = 0$ such that each $\mathcal{F}_i/\mathcal{F}_{i+1}$ is an $\mathcal{O}_Y$-module, To show surjective on the right use problem 5.]

2. (*) (Hartshorne Ex 6.11 in II.6) The Grothendieck group of a nonsingular curve. Let X be a nonsingular curve (integral noetherian separated) over an algebraically closed field k. We will show that $K(X) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}$ in several steps.

- **(a)** For any divisor $D = \sum n_i P_i$ on X, let $\psi(D) = \sum n_i \gamma(k(P_i)) \in K(X)$ where $k(P_i)$ is the skyscraper sheaf k at P_i and 0 everywhere. If D is an effective divisor, let $\mathcal{O}_D$ the the structure sheaf of the associated subscheme of codimension 1 and show that $\psi(D) = \gamma(\mathcal{O}_D)$. Then show that for any D, $\psi(D)$ depends only on the linear equivalence class of D so ψ defines a homomorphism $\psi : Cl(X) \to K(X)$.
- **(b)** For any coherent sheaf $\mathcal{F}$ on X show that there exists a locally free sheaves $\mathcal{E}_0$ and $\mathcal{E}_1$ and an exact sequence $0 \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$. Let r_0 =rank $\mathcal{E}_0$, r_1 =rank $\mathcal{E}_1$, and define $\det \mathcal{F} = (\wedge^{r_0} \mathcal{E}_0) \otimes (\wedge^{r_1} \mathcal{E}_1)^{-1} \in \mathrm{Pic}(X)$. Here $\wedge$ denotes the exterior power. Show that $\det \mathcal{F}$ is independent of the resolution chosen and that it gives an homomorphism $\det : K(X) \to \mathrm{Pic}(X)$. Finally show that if D is a divisor then $\det(\psi(D)) = \mathcal{O}(D)$
- **(c)** If $\mathcal{F}$ is any coherent sheaf of rank r, show that there is a divisor D on X and an exact sequence $0 \to \mathcal{O}(D)^{\oplus r} \to \mathcal{F} \to \mathcal{T} \to 0$, where $\mathcal{T}$ is a torsion sheaf. Conclude that if $\mathcal{F}$ is sheaf of rank r, then $\gamma(\mathcal{F}) - r\gamma(\mathcal{O}_X) \in Im\psi$
- **(d)** Using the maps ψ, det, rank and $1 \mapsto \gamma(\mathcal{O}_X)$ from $\mathbb{Z} \to K(X)$ show that $K(X) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}$

3. (Hartshorne Ex 6.12 in II.6) Let X be a complete nonsingular curve. Show that there is a unique way to define the *degree* of any coherent sheaf on X, $\deg \mathcal{F} \in \mathbb{Z}$ such that:

- **(1)** If D is a divisor, $\deg \mathcal{O}(D) = \deg D$

(2) If $\mathcal{F}$ is a torsion sheaf (meaning a sheaf whose stalk at the generic point is zero), then $\det \mathcal{F} = \sum_{P \in X}$ length $(\mathcal{F}_p)$ and

(3) If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence, then $\deg \mathcal{F} = \deg \mathcal{F}' + \deg \mathcal{F}''$.

4. (Ex 6.6 in Hartshorne I.6) Automorphisms of $\mathbb{P}^1$. Think of $\mathbb{P}^1$ as $\mathbb{A}^1 \cup \{\infty\}$. Then we define a *fractional linear transformation* of $\mathbb{P}^1$ by sending $x \mapsto \frac{ax+b}{cx+d}$ for $a, b, c, d \in k$ and $ad - bc \neq 0$.

(a) Show that a fractional linear transformation induces an automorphism of $\mathbb{P}^1$ (i.e an isomorphism of $\mathbb{P}^1$ with itself). We denote the group of all these fractional linear transformations by $PGL(1)$

(b) Let $\operatorname{Aut} \mathbb{P}^1$ denote the group of all automorphisms of $\mathbb{P}^1$. Show that $\operatorname{Aut} \mathbb{P}^1 \cong \operatorname{Aut} k(x)$ the group of k-automorphisms of the field $k(x)$

(c) Now show that every automorphism of $k(x)$ is a fractional linear transformation and deduce that $PGL(1) \to \operatorname{Aut} \mathbb{P}^1$ is an isomorphism.

5. (Ex 5.15 in Hartshorne II.5) Extension of Coherent sheaves. We will prove the following theorem in several steps Let X be a noetherian scheme, let U be an open subset, and let $\mathcal{F}$ be a coherent sheaf on U. Then there is a coherent sheaf $\mathcal{F}'$ on X such that $\mathcal{F}'|_U \cong \mathcal{F}$.

(a) On a notherian affine scheme, every quasi-coherent sheaf is the union of its coherent subsheaves. We say a sheaf $\mathcal{F}$ is the union of its subsheaves $\mathcal{F}_a$ if for every open set U, the group $\mathcal{F}(U)$ is the union of the groups $\mathcal{F}_a(U)$

(b) Let X be an affine noetherian shceme, let U be an open subset, and $\mathcal{F}$ coherent on U. Then there exists a coherent sheaf $\mathcal{F}'$ on X with $\mathcal{F}'|_U \cong \mathcal{F}$. [Hint: Let $i : U \to X$ be the inclusion map. Show that $i_*\mathcal{F}$ is quasi-coherent, then use (a).]

(c) Let $X, Y, \mathcal{F}$ as in (b), suppose furthermore we are given a quasi-coherent sheaf $\mathcal{G}$ on X such that $\mathcal{F} \subseteq \mathcal{G}|_U$. Show that we can find $\mathcal{F}'$ a coherent subsheaf of $\mathcal{G}$ with $\mathcal{F}'|_U \cong \mathcal{F}$. [Hint: Use the same method, but replace $i_*\mathcal{F}$ by $\rho^{-1}(i_*\mathcal{F})$ where ρ is the natural map $\mathcal{G} \to i_*\mathcal{G}$.]

(d) Now let X be any noetherian scheme, U an open subset, $\mathcal{F}$ a coherent sheaf on U, and $\mathcal{G}$ a quasi-coherent sheaf on X such that $\mathcal{F} \subset \mathcal{G}|_U$. Show that there is a coherent subsheaf $\mathcal{F}' \subset \mathcal{G}$ on X with $\mathcal{F}'|_U \cong \mathcal{F}$. [Hint: Cover X with open affines and extend over one of them at a time.]

(e) As an extra corollary, show that on a noetherian scheme, any quasi-coherent sheaf $\mathcal{F}$ is the union of its coherent subsheaves. [Hint: if s is a section of $\mathcal{F}$ over an open set U, apply d to the subsheaf of $\mathcal{F}|_U$ generated by s.]

6. (Ex 7.2 in "Algebraic geometry 3" by Ueno) Let X be a proper algebraic scheme over an algebraically closed field k, and let V be the sub-vector space of $\Gamma(X, \mathcal{L})$ spanned by $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$ of an invertible sheaf $\mathcal{L}$. Assume that V generates the invertible sheaf $\mathcal{L}$. That is, assume that the natural homomorphism $\psi : \mathcal{O}_X \otimes_k V \to \mathcal{L}$ is surjective,. (We can also express this condition as follows: $D_+(s_i) = \{x \in X | s_j(x) \neq 0\}, i = 0, 1, \ldots, n$ form an open covering of X. Namely, for an arbitrary point x of X, one can find s_j satisfying $s_j(x) \neq 0$). Then show that ϕ induces the morphism $\phi : X \to \mathbb{P} \cong \mathbb{P}^n_k$ and that this morphism ϕ is a closed immersion if and only if the following two conditions are satisfied:

(i) Points in X are separated by elements in V. That is, for any two closed points x and y in X, there are s and t in V such that $s \in \mathfrak{m}_x\mathcal{L}_x, s \notin \mathfrak{m}_y\mathcal{L}_y$ and $t \in \mathfrak{m}_y\mathcal{L}_y$.

(ii) Elements of V separate tangent vectors at each point of X. Namely, for an arbitrary closed point $x \in X$, the set $\{s \in V | s_x \in \mathfrak{m}_x\mathcal{L}_x\}$ spans the vector space $\mathfrak{m}_x\mathcal{L}_x/\mathfrak{m}_x^2\mathcal{L}_x$

Note that in (i) and (ii) , $\mathfrak{m}_x$ and $\mathfrak{m}_y$ are the maximal ideals of $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X,y}$ at x and y respectively, and let $\mathcal{L}_x$ and $\mathcal{L}_y$ are the stalks of $\mathcal{L}$ at x and y respectively.

7. Let X be a noetherian separated integral normal scheme X. Let D be a Weil divisor. Prove that the sheaf $\mathcal{O}_X(D)$ is quasi-coherent.