Homework 2

Math 607 (section 0101), Spring 2022

Due Wednesday, February 23

You are encouraged to think about problems marked with a (*), but they are not to be handed in.

1. (a) Let $X = \mathbb{A}^1_k = \operatorname{Spec} k[t]$ for an algebraically closed field k. Let $M = k[t]/(t)$ and $\mathcal{F} = \widetilde{M}$. Calculate the rank of $\mathcal{F}$ at all points i.e. at both the closed points and the generic point.

(b) Let $X = \mathbb{A}^2_k = \operatorname{Spec} k[s,t]$ for an algebraically closed field k. Consider the ideal $I = (s,t)$ and the sheaf $\mathcal{F} = \widetilde{I}$. Calculate the rank of $\mathcal{F}$ at all closed points.

(c) Let $X = \operatorname{Spec} k[x]/(x^2)$ and $M = k$. Show that the sheaf $\mathcal{F} = \widetilde{M}$ has constant rank 1 but it is not locally free.

2. (Hartshorne Ex 5.1 in II.5) Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of finite rank. We define *dual* of $\mathcal{E}$, denote $\check{\mathcal{E}}$. to be the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$.

(a) Show that $(\check{\mathcal{E}})\check{} \cong \mathcal{E}$

(b) For any $\mathcal{O}_X$-module $\mathcal{F}$, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \cong \check{\mathcal{E}} \otimes \mathcal{F}$

(c) For any $\mathcal{O}_X$-modules $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E} \otimes \mathcal{F}, \mathcal{G}) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{G}))$

(d) (Projection formula) If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, if $\mathcal{F}$ is an $\mathcal{O}_X$-module and $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank, then there is a natural isomorphism $f_*(\mathcal{F} \otimes f^*\mathcal{E}) \cong f_*(\mathcal{F}) \otimes \mathcal{E}$

(e) Let $\mathcal{E}$, $\mathcal{F}$ be locally free $\mathcal{O}_X$-modules of finite ranks m and n respectively. Show that $\mathcal{E} \otimes \mathcal{F}$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ are locally free of rank mn and $\check{\mathcal{E}}$ is locally free of rank m.

3. (Ex 5.3 in Hartshorne I.5) Multiplicities. Let $Y \subset \mathbb{A}^2_k$ for algebraically closed k be a curve defined by the equation $f(x,y) = 0$. Let $P = (a,b)$ be a closed point of $\mathbb{A}^2$. Make a linear change of coordinates so that P becomes the point $(0,0)$. Then write f as a sum $f = f_0 + f_1 + \ldots + f_d$ where f_i is homogeneous polynomial of degree i in x and y. Then we define the *multiplicity* of P on Y, denoted $\mu_P(Y)$, to be the least r such that $f_r \neq 0$. (Note $P \in Y \iff \mu_P(Y) > 0$.) The linear factors of f_r are called *tangent directions* at P.

(a) Show that $\mu_P(Y) = 1 \iff P$ is a nonsingular point.

(b) Find the multiplicity of each of the singular points in Ex 5.1 i.e. the curves in $\mathbb{A}^2$ cut out by (a) $x^2 = x^4 + y^4$; (b) $xy = x^6 + y^6$; (c) $x^3 = y^2 + x^4 + y^4$; (d) $x^2y + xy^2 = x^4 + y^4$.

4. (Ex 5.4 in Hartshorne I.5) Intersection multiplicity. If $Y, Z \subset \mathbb{A}^2_k$ over an algebraically closed field k are two distinct curves, given by equations $f = 0$ and $g = 0$ and if $P \in Y \cap Z$ is a closed point, we define the *intersection multiplicity* $(Y \cdot Z)_P$ of Y and Z at P to be the length of the $\mathcal{O}_P$ module $\mathcal{O}_P/(f,g)$.

(a) Show that $(Y \cdot Z)_P$ is finite and $(Y \cdot Z)_P \geq \mu_P(Y)\mu_P(Z)$.

(b) IF $P \in Y$, show that for almost all lines L through P (i.e. all but a finite number), $(L \cdot Y)_P = \mu_P$

(c) If Y is a curve of degree d in $\mathbb{P}^2$ and L is a line in $\mathbb{P}^2$ with $L \neq Y$, show that $(L \cdot Y) = d$. Here $(L \cdot Y) = \sum (L \cdot Y)_P$ taken over all closed points $P \in L \cap Y$, where $(L \cdot Y)_P$ is defined for a suitable affine cover of $\mathbb{P}^2$.

5. (Ex 5.9 in Hartshorne I.5) Let $f \in k[x, y, z]$ $(k = \bar{k})$ be a homogeneous polynomial, let $Y = V(f) \subset \mathbb{P}^2$ be the algebraic set defined by f, and suppose that for every closed point $P \in Y$ at least one of $\partial f/\partial x(P), \partial f/\partial y(P), \partial f/\partial z(P)$ is nonzero. Show that f is irreducible and hence Y is a nonsingular variety. Hint: Use that in $\mathbb{P}^2$ any two curves have non-empty intersection.

6. (Ex 5.10 in Hartshorne I.5) For a closed point P on a variety X over algebraically closed field, let $\mathfrak{m}$ be the maximal ideal of the local ring $\mathcal{O}_P$. We define the *Zariski tangent space* $T_P(X)$ of X at P to be the dual k-vector space of $\mathfrak{m}/\mathfrak{m}^2$. Show the following:

(a) For any closed point $P \in X$, $\dim T_P(X) \geq \dim X$ with equality if and only if P is nonsingular.

(b) For any morphism $\phi : X \to Y$ of varieties, there is a natural induced k-linear map $T_P(\phi) : T_P(X) \to T_{\phi(P)}(Y)$

(c) If ϕ is the vertical projection of the parabola $x = y^2$ onto the x-axis, show that the induced map $T_0(\phi)$ of the tangent spaces at the origin is the zero map.

7. Rational functions on a scheme X are the equivalent classes of section of $\mathcal{O}_X$ over dense and open subsets of X. So the ring of rational functions form a ring denoted by $R(X)$.

(a) If X is irreducible, explain why $R(X) = \mathcal{O}_{X,x}$ where x is the generic point of X.

(b) If X is irreducible, explain why for every non-empty open subset $U \subset X$ we have $R(X) = R(U)$.

(c) If $X = \operatorname{Spec} A$ and A is noetherian, let $S = A/\cup p_i$ where the union if over all minimal primes $p_i \subset A$ (corresponding to irreducible components of X). Show that $R(X) \cong A_S \cong \prod A_{p_i}$.

(d) If X is irreducible and noetherian show that sheaf of rational functions defined in class is constant and in particular when X is integral it is the constant field $K(X)$.