Homework 1

Math 607 (section 0101), Spring 2022

Due Wednesday, February 16

You are encouraged to think about problems marked with a (*), but they are not to be handed in.

1. (Hartshorne Ex 5.7 in II.5) Let X be a noetherian scheme and $\mathcal{F}$ a coherent sheaf (i.e. locally $\mathcal{F} = \tilde{M}$ for a finitely generated module M.) Prove the following statements

- **(a)** If the stalk $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$ for one point $x \in X$, then there is an open neighborhood U of x such that $\mathcal{F}|_U$ is free.
- **(b)** $\mathcal{F}$ is locally free iff the stalks $\mathcal{F}_x$ are free for all points $x \in X$
- **(c)** $\mathcal{F}$ is invertible (i.e. locally free of rank 1) iff there is a coherent sheaf $\mathcal{G}$ on X such that $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{O}_X$

2. (Hartshorne Ex 5.8 in II.5) Let X be a noetherian scheme and $\mathcal{F}$ a coherent sheaf. Consider the function $\phi : X \to \mathbb{R}$ defined as $\phi(x) = \dim_{k(x)} \mathcal{F}_x \otimes k(x)$. We call $\phi(x)$ the rank of $\mathcal{F}$ at x. Prove the following statements about the rank function

- **(a)** The function ϕ is semi-continuous i.e. the sets $\{x \in X | \phi(x) \geq n\}$ are closed in X.
- **(b)** If $\mathcal{F}$ is locally free and X is connected, then ϕ is a constant function
- **(c)** Conversely if X is reduced and ϕ is a constant function, then $\mathcal{F}$ is a locally free shear.

3. Consider the cuspidal cubic $X = \operatorname{Spec} k[x, y](/y^2 - x^3)$.

- **(a)** Calculate the normalization $\tilde{X}$ of X.
- **(b)** Calculate the fibre of the cusp at the origin of X inside the normalization $\tilde{X}$

4. Let R be a commutative ring and M is an R-module. Prove

- **(a)** M is flat R-module iff its localizations
  - **(a)** M is flat R-module iff its localizations $M_{\mathfrak{p}}$ at all primes $\mathfrak{p}$ is a flat $R_{\mathfrak{p}}$-module
  - **(b)** M is flat iff $\operatorname{Tor}_1(M, R/I) = 0$ for all ideals $I \subset R$
  - **(c)** Assume R is a noetherian ring, then M is flat iff $\operatorname{Tor}_1(M, R/\mathfrak{p}) = 0$ for all primes $\mathfrak{p}$
  - **(d)** Assume R is noetherian and M is finitely generated, then M is flat iff $\operatorname{Tor}_1(M, k(\mathfrak{p})) = 0$ for all max ideals $\mathfrak{p}$

5. Consider $A = k[x, y, x]/(z^2 - xy)$, $X = \operatorname{Spec} A$, and the point $\mathfrak{p} = (x, z) \in X$.

- **(a)** Prove $A_{\mathfrak{p}} \cong k[y, z]_(z)$
- **(b)** Prove that $A_{\mathfrak{p}}$ is regular ring of dimension 1.

6. Prove the isomorphism in the localization $(k[x, y]/(xy - 1))_{(x-1,y-1)} \cong k[x - 1]_{(x-1)}$. Note that we are localizing at prime ideals i.e. at points!

7. (*) (Hartshorne Ex 3.22 in II.3) Let $f : X \to Y$ be a dominant morphism of integral schemes of finite type over a field k.

(a) Let Y' be a closed irreducible subset of Y, whose generic point η' is contained in $f(X)$. Let Z be any irreducible component of $f^{-1}(Y')$ such that $\eta' \in f(Z)$ and show that $\operatorname{codim}(Z, X) \leq \operatorname{codim}(Y', Y)$

(b) Let $e = \dim X - \dim Y$ be the relative dimension of X over Y. For any point $y \in f(X)$ show that every irreducible component of the fibre X_y has dimension $\geq e$ (Hint: Let $Y' = V(y)$ and use (a) and the fact that for integral schemes W of finite type over k we have $\dim W = tr.d.K(W/)/k$.

(c) Show that there is a dense open subset $U \subset X$ such that for any $y \in f(U)$, $\dim U_y = e$. Hint: First reduce to the case where X and Y are affine, say $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Then A is finitely generated B-algebra. Take $t_1, \dots, t_e \in A$ which form a transcendence base of $K(X)$ over $K(Y)$ and let $X_1 = \operatorname{Spec} B[t_1, \dots, t_e]$. Then X_1 is isomorphic to affine e-space over Y an the morphism $X \to X_1$ is generically finite. Now use Ex 3.7

(d) Going back to out original morphism $f : X \to Y$ for any integer h, let E_h be the set of points $x \in X$ such that $y = f(x)$, there is an irreducible component Z of the fibre X_y, containing x, and having $\dim Z \geq h$. Show that (1) $E_e = X$ (use (b) above); (2) if $h > e$, then E_h is not dense in X (use (c) above); (3) E_h is closed, for all h (use induction on $\dim X$).

(e) Prove the following theorem of Chevalley - see Cartan and Chevalley [1, expose 8]. For each integer h, let C_h be the set of points $y \in Y$ such that $\dim X_y = h$. Then the subsets C_h are constructible, and C_e contains dense subset of Y.

8. (*) (Hartshorne Ex 10.1 in III.10) Given a nonperfect field, smooth and regular are not equivalent. For example. let k_0 be a field of characteristic $p > 0$, let $k = k_0(t)$ and let $X \subset \mathbb{A}^2_k$ be the curve defined by $y^2 = x^p - t$. Show that every local ring of X is a regular local ring but X is not smooth over k.