HW 1

1. Let $I < k[x_1, \dots, x_n]$ be a nonzero ideal (k is algebraically closed field) with the reduced Gröbner basis G with respect to the lexicographic ordering $x_1 > x_2 > \cdots > x_n$. Show that the following statements are equivalent:
   (a) The quotient $k[x_1, \dots, x_n]/I$ is a finite dimensional vector space.
   (b) $I \cap k[x_i] \neq 0$ for any $i = 1, \dots, n$.
   (c) For each $i = 1, \dots, n$, there is a $g_i \in G$ with leading term $\mathrm{LT}(g_i) = x_i^{n_i}$ for some $n_i \geq 1$.
   (d) $V(I)$ is a finite set of points.
   (Hint: For $a \Rightarrow b$ use the inclusion $k[x_i]/(I \cap k[x_i]) \hookrightarrow k[x_1, \dots, x_n]/I$. For $b \Rightarrow c$ argue that some $\mathrm{LT}(g_i)$ divides the leading term of a generator of $I \cap k[x_i]$. Show $c \Rightarrow a$ and $b \Rightarrow d$. For $d \Rightarrow b$ if $V(I) = \{p_1, \dots, p_m\}$ and a_k is the i-th coordinate of p_k then show that $\prod_{k=1}^{m}(x_i - a_k) \in I(V(I)) = \sqrt{I}$.)

2. Let S be the rational normal cone in $\mathbb{C}^3$ parameterized by $\Phi\colon \mathbb{C}^2 \to \mathbb{C}^3$
$$\Phi(s,t) = (s^2, t^2, st).$$
   a) Show that S is an algebraic set by proving $S = V(I)$ where I is the ideal $(z^2 - xy) < \mathbb{C}[x, y, z]$.
   b) Show that S is irreducible.
   c) Show that $I(S) = (z^2 - xy)$.
   d) Is $\Phi\colon \mathbb{C}^2 \to S$ a regular map? Is it an isomorphism?
   e) Consider the morphism $\Psi\colon \mathbb{C}^2 \to S$ determined by the $\mathbb{C}$-algebra homomorphism
$$\frac{\mathbb{C}[x, y, z]}{(z^2 - xy)} \to \mathbb{C}[u, v]$$
   that maps $x \mapsto u$, $z \mapsto uv$ and $y \mapsto uv^2$. Describe $\Psi^{-1}(p)$ for any point $p \in S$.

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