Exam 4 Spring 2021

$\boxed{1a}$ $\lim_{n\to\infty}(4e^{-n}-3e^{\frac{1}{n}}) = 4\lim_{n\to\infty}e^{-n} - 3\lim_{n\to\infty}e^{\frac{1}{n}} = 4\cdot 0 - 3\cdot 1 = -3$

$\boxed{1b}$ Recall the geometric series: $\sum_{n=m}^{\infty} cr^n = \frac{1\text{st term}}{1-r} = \frac{cr^m}{1-r}$, so

$$\sum_{n=2}^{\infty}\frac{2^{n+2}}{5(3^n)} = \sum_{n=2}^{\infty}\frac{2^2\cdot 2^n}{5\cdot 3^n} = \sum_{n=2}^{\infty}\frac{4}{5}\cdot\left(\frac{2}{3}\right)^n \underset{m=2}{\overset{c=\frac{4}{5},\ r=\frac{2}{3}}{=\!=\!=}} \frac{\frac{4}{5}\cdot\left(\frac{2}{3}\right)^2}{1-\frac{2}{3}} = \frac{16}{15}$$

$\boxed{2a}$ Note that $-1\le\cos(x)\le 1 \Rightarrow 0\le\cos^2 x\le 1$, so

$$\sum_{n=1}^{\infty}\frac{\cos^2 n}{n^{3/2}} \underset{\substack{\text{comparison}\\ \text{test}}}{\le} \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$$ converges by p-series test with $p=\frac{3}{2}>1$

$\boxed{2b}$ Apply ratio test

$$\lim_{n\to\infty}\left|\frac{\frac{((n+1)!)^2}{(2(n+1))!}}{\frac{(n!)^2}{(2n)!}}\right| = \lim_{n\to\infty}\frac{\frac{((n+1)\cdot\cancel{n!})^2}{(2n+2)(2n+1)\cancel{(2n)!}}}{\frac{\cancel{(n!)^2}}{\cancel{(2n)!}}} = \lim_{n\to\infty}\frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1$$

Hence $\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}$ converges

$\boxed{2c}$ If you try, you will see that ratio and root tests do NOT work!

We shall use the n-th term test: since

$$\lim_{n\to\infty}\left(\frac{n^2}{n^2+n}\right)^4 = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^4 = \left(\frac{1}{1+0}\right)^4 = 1 \neq 0,$$ $\sum_{n=3}^{\infty}\left(\frac{n^2}{n^2+n}\right)^4$ can not possibly be convergent in the first place, so it must be divergent

$\boxed{3a}$ By alternating series test, $\left\{\frac{1}{4n}\right\}_{n=1}^{\infty}$ is positive, decreasing, $\lim_{n\to\infty}\frac{1}{4n}=0$

so $\sum_{n=1}^{\infty}(-1)^n\frac{1}{4n}$ converges

On the other hand, $\sum_{n=1}^{\infty}\left|(-1)^n\frac{1}{4n}\right| = \sum_{n=1}^{\infty}\frac{1}{4n}$ is divergent by p-series test with $p=1$. Hence $\sum_{n=1}^{\infty}(-1)^n\frac{1}{4n}$ converges conditionally

Recall $\left|\sum_{n=1}^{\infty}(-1)^n a_n - \sum_{n=1}^{j}(-1)^n a_n\right| < a_{j+1}$, we want

$\frac{1}{4(j+1)} = a_{j+1} < \frac{1}{20} \Rightarrow j+1 > 5 \Rightarrow j > 4 \Rightarrow j = 5$

$\boxed{3b}$ $f'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{n}{3^n}x^{2n}\right) = \sum_{n=0}^{\infty}\frac{d}{dx}\left(\frac{n}{3^n}x^{2n}\right) = \sum_{n=1}^{\infty}\frac{2n^2}{3^n}x^{2n-1}$

use ratio test:

$\lim_{n\to\infty}\left|\frac{\frac{2(n+1)^2}{3^{n+1}}x^{2n+1}}{\frac{2n^2}{3^n}x^{2n-1}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{3n^2}x^2\right| = \frac{1}{3}|x|^2 \leq 1 \Rightarrow |x| \leq \sqrt{3}$

therefore $R = \sqrt{3}$

$\boxed{4a}$. Recall: $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$, so

$g(x) = x^2\cos(3x) = x^2\left(1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \cdots\right) = x^2 - \frac{3^2x^4}{2!} + \frac{3^4x^6}{4!} - \cdots$

$= x^2\left(\sum_{n=0}^{\infty}\frac{(-1)^n(3x)^{2n}}{(2n)!}\right) = x^2\left(\sum_{n=0}^{\infty}\frac{(-1)^n 3^{2n}\cdot x^{2n}}{(2n)!}\right)$

$\overset{\text{distribute}}{=\!=} \sum_{n=0}^{\infty}\frac{(-1)^n 3^{2n}\cdot x^{2n}\cdot x^2}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n 3^{2n}}{(2n)!}x^{2n+2}$

To get $P_4(x)$, we want those terms with degree $2n+2 \leq 4 \Rightarrow n \leq 1$

so $P_4(x) = \sum_{n=0}^{1}\frac{(-1)^n 3^{2n}}{(2n)!}x^{2n+2} = x^2 - \frac{9x^4}{2}$

$\boxed{4b}$ Lagrange remainder theorem: $r_n(x) = \frac{f^{(n+1)}(t_x)}{(n+1)!}x^{n+1}$ for some t_x between 0 and x

Note $|f^{(n+1)}(x)| \leq 2^{n+1}$ [Reason: $\sin(2x)' = 2\cos(2x)$, $\cos(2x)' = -2\sin(2x)$

so each derivative gets a factor of 2]

Hence $|r_n(x)| \leq \frac{2^{n+1}}{(n+1)!}|x|^{n+1} \xrightarrow{n\to\infty} 0$

$\boxed{5.}$ $z = Re^{i\theta}$, $r = R^{\frac{1}{n}}$, the n-th roots are $z^{\frac{1}{n}} = r\left(\cos\left(\frac{\theta+k\cdot 2\pi}{n}\right) + i\sin\left(\frac{\theta+k\cdot 2\pi}{n}\right)\right)$

for $k = 0, 1, 2, \cdots, n-1$

In our case

$$9i = 9e^{i\frac{\pi}{2}}, \quad n=2, \quad R=9, \quad r = 9^{\frac{1}{2}} = 3, \quad \theta = \frac{\pi}{2}$$

so the square roots are

$$\begin{cases} z_0 = 3\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) = 3\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) & \leftarrow k=0 \\ z_1 = 3\left(\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right) = 3\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) & \leftarrow k=1 \end{cases}$$

Plot them on the complex plane