Exercise:

Suppose $f(x)=x^2e^{x^3}=\sum_{n=0}^{\infty}a_nx^n$, what is

a_4, a_7, a_{28}, a_{29}

Solution: Note that $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots=\sum_{n=0}^{\infty}\frac{x^n}{n!}$

so $e^{x^3}=1+(x^3)+\frac{(x^3)^2}{2!}+\frac{(x^3)^3}{3!}+\frac{(x^3)^4}{4!}+\cdots=\sum_{n=0}^{\infty}\frac{(x^3)^n}{n!}$

and $x^2e^{x^3}=x^2\left(1+x^3+\frac{x^6}{2!}+\frac{x^9}{3!}+\frac{x^{12}}{4!}+\cdots\right)=x^2\sum_{n=0}^{\infty}\frac{x^{3n}}{n!}$

$$=x^2+x^5+\frac{x^8}{2!}+\frac{x^{11}}{3!}+\frac{x^{14}}{4!}+\cdots=\sum_{n=0}^{\infty}\frac{x^{3n+2}}{n!}=\sum_{n=0}^{\infty}\frac{1}{n!}\cdot x^{3n+2}$$

so this power series only have nonzero coefficients in front of terms of degree $3n+2$, ($n\geq 0$ integers)

Hence
$$\left.\begin{array}{l}3n+2=4\Rightarrow n=\frac{2}{3}\\ 3n+2=7\Rightarrow n=\frac{5}{3}\\ 3n+2=28\Rightarrow n=\frac{26}{3}\end{array}\right\}\ n \text{ not an integer} \Rightarrow \left\{\begin{array}{l}a_4=0\\ a_7=0\\ a_{28}=0\end{array}\right.$$

$3n+2=29\Rightarrow n=9 \Rightarrow a_{29}$ is the coefficient of term $a_{29}x^{29}$

$$a_{29}x^{29} = a_{29}x^{3\cdot 9+2} = \frac{1}{9!}x^{3\cdot 9+2}$$

(here $n=9$)

so $a_{29}=\frac{1}{9!}$