Formula sheet

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1 Derivatives

Summation(Subtraction): $(af(x) \pm bg(x))' = af'(x) \pm bg'(x)$, for example $(e^x - 2\sin x)' = (1 \cdot e^x + (-2) \cdot \sin x)' = 1 \cdot (e^x)' + (-2)(\sin x)' = (e^x)' - 2(\sin x)'$, or simply $(e^x - 2\sin x)' = (e^x)' - 2(\sin x)' = e^x + 2\cos x$, if we take $b = 0$, then $(af(x))' = af'(x)$

Product: $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$, for example $(e^x \sin x)' = (e^x)' \sin x + e^x (\sin x)' = e^x \sin x + e^x \cos x$

Composition: $[f(g(x))]' = f'(g(x))g'(x)$, for example $\left[\ln(1+5x^2)\right]'$, let $f(x) = \ln x, g(x) = 1 + 5x^2$, then $f'(x) = \frac{1}{x}, g'(x) = 10x$, $\left[\ln(1+5x^2)\right]' = [f(g(x))]' = f'(g(x))g'(x) = \frac{1}{g(x)} g'(x) = \frac{1}{1+5x^2} \cdot 10x = \frac{10x}{1+5x^2}$

Quotient: $\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$, notice quotient is a product with composition $\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}$

Derivative of some elementary functions

$(x^n)' = nx^{n-1}$, $(e^x)' = e^x$, $(\ln x)' = \frac{1}{x} = x^{-1}$,$(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $(\tan x)' = \sec^2 x$

2 Integrations

Summation(Subtraction): $\int (af(x) \pm bg(x))\,dx = a \int f(x)dx \pm b \int g(x)dx$, for example $\int (e^x + 2\cos x)\,dx = \int e^x dx + 2 \int \cos x dx = e^x + 2\sin x + C$. Note that adding constant C is to cover all antiderivatives, if we take $b = 0$, then $\int af(x)dx = a \int f(x)dx$

$\int f(g(x))g'(x)dx \overset{u=g(x),du=g'(x)dx}{=\!=\!=} \int f(u)\,du$, this is integration by substitution, for example

$\int e^{\sin x} \cos x dx \overset{u=\sin x,du=\cos x dx}{=\!=\!=} \int e^u du = e^u + C = e^{\sin x} + C$

$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$, this is integration by parts, for example $\int xe^x dx = \int x(e^x)'dx = xe^x - \int e^x x' dx = xe^x - \int e^x dx = xe^x - e^x + C$

Integrations of some elementary functions

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ when $n + 1 \neq 0$, $\int x^{-1} dx = \int \frac{1}{x} dx = \ln x + C$, $\int e^x dx = e^x + C$, $\int \sin x dx = -\cos x + C$, $\int \cos x dx = \sin x + C$, $\int \sec^2 x dx = \tan x + C$

3 Trigonometry identities

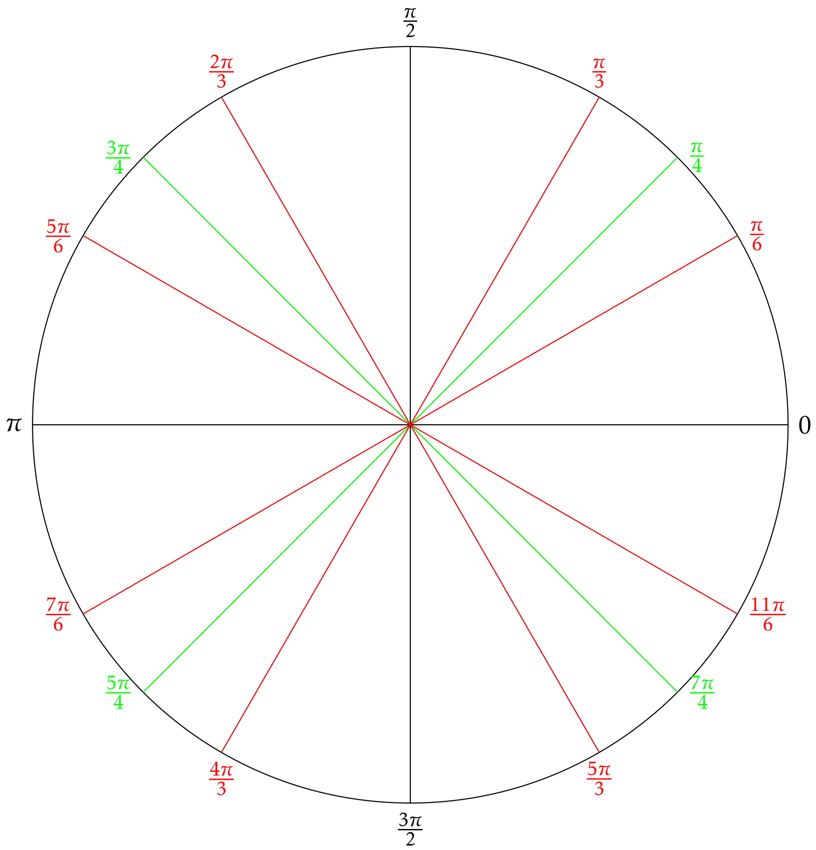
$\cos^2 x + \sin^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$\sin(-x) = -\sin x$, $\cos(-x) = \cos x$, $\tan(-x) = -\tan x$, $\tan x = \frac{\sin x}{\cos x}$, $\sec x = \frac{1}{\cos x}$,

4 Trigonometry table

θ	0	$\frac{\pi}{6} = 30°$	$\frac{\pi}{4} = 45°$	$\frac{\pi}{3} = 60°$	$\frac{\pi}{2} = 90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$\times$



5 Approximation to definite integrals

Given $a = a_0, b = a_n$, and divide interval (a, b) into n equal parts $(a_0, a_1), (a_1, a_2), \cdots, (a_{n-2}, a_{n-1}), (a_n, a_n)$ with $\Delta x = a_i - a_{i-1}$ being the length of each subinterval, then $a_i = a_{i-1} + \Delta x$

Trapezoidal rule: $\int_a^b f(x)dx \approx [f(a_0) + 2f(a_1) + 2f(a_2) + \cdots + 2f(a_{n-1}) + f(a_n)] \frac{\Delta x}{2}$

6 Method of Integrating factors

Always starts with a DE(differential equation) of the form $y' + a(t)y = b(t)$. Note that if your DE is $t^2y' + y = t^3$, then first you need to divide t^2 on both sides, then you would have $y' + \frac{1}{t^2}y = t$ with $a(t) = \frac{1}{t^2}, b(t) = t$

Then take any antiderivative $A(t) = \int a(t)dt$, such that $A'(t) = a(t)$

Then we get the integrating factor $e^{A(t)}$, for example if $a(t) = 2$, then $A(t)$ could be $2t$, then $e^{A(t)} = e^{2t}$

Then we have $\left[ye^{A(t)}\right]' = b(t)e^{A(t)}$

Then we integrate on both sides to get $ye^{A(t)} = \int ye^{A(t)}dt = \int b(t)e^{A(t)}dt$

At last, we divide $e^{A(t)}$ on both sides(which is equivalent to multiplying $e^{-A(t)}$ on both sides), we get $y = e^{-A(t)} \int b(t)e^{A(t)}dt$

For example, suppose the DE we have is $y' + 2y = t$

First we identify $a(t) = 2, b(t) = t$
Find one antiderivative of $a(t)$, $A(t) = \int 2dt = 2t$, notice that the reason for omitting C is that we only need to find one such antiderivative
The integrating factor is $e^{A(t)} = e^{2t}$
We have $\left[ye^{2t}\right]' = te^{2t}$
We have $ye^{2t} = \int te^{2t}dt = \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + C$
Dividing e^{2t}(or equivalently, multiplying e^{-2t}), we have $y = \frac{1}{2}t - \frac{1}{4} + Ce^{-2t}$

7 Geometric series and integral test

A geometric series is $a + ar + ar^2 + ar^3 + \cdots$, where $|r| < 1$, the sum is given by $\frac{a}{1-r}$
Suppose $f(x)$ is a continuous, positive, decreasing function on $x \geq n$, then $\int_n^\infty f(x)dx$ is convergent if and only if $\sum_{k=n}^{\infty} f(k)$ is convergent, which doesn't mean the have the same value!

8 Taylor polynomials

The n-th Taylor polynomial of $f(x)$ at $x = a$ is the polynomial $p_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$, note that $p_n(a) = f(a), p_n'(a) = f'(a), p_n''(a) = f''(a), \cdots, p_n^{(n)}(a) = f^{(n)}(x)$, for example $(e^x)^{(n)} = e^x$, thus the n-th Taylor polynomial of e^x at $x = 0$ is $p_n(x) = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n$

9 Random variable and probability

Suppose X is a continuous random variable taking values within A and B, where A can be $-\infty$ and B can be ∞, we define the probability through a density function $Pr(a \leq X \leq b) = \int_a^b f(x)dx, A \leq a \leq b \leq B$ where f is called the probability density function satisfying
(1) $f(x) \geq 0$ for $A \leq x \leq B$(since the probability has to be nonnegative)
(2) $\int_A^B f(x)dx = 1$(since the integral of all possibilities should be 1)
We also define cumulative distribution function $F(x) := Pr(A \leq X \leq x) = \int_A^x f(t)dt$(here use t because x is already taken), then we have $F'(x) = f(x)$ and thus by fundamental theorem of calculus we have $Pr(a \leq X \leq b) = \int_a^b f(x)dx = F(b) - F(a)$
So given a probability density function of a random variable X, we can find its cumulative distribution function by definition, given its cumulative distribution function, we can get its probability distribution function by taking derivative
Let X be a continuous random variable whose possible values lie between A and B, and let $f(x)$ be the probability density function for X. Then the expected value(or mean) of X is defined to be $E(X) := \int_A^B xf(x)dx$, and variance to be $Var(X) = \int_A^B x^2 f(x)dx - E(X)^2$
Exponential distribution: $X \sim E(k)$, pdf: $f(x) = ke^{-kx}(k > 0)$ for $x \geq 0$, $E(X) = \frac{1}{k}$, $Var(X) = \frac{1}{k^2}$
Normal distribution: $X \sim N(\mu, \sigma^2)$, pdf: $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, $\sigma > 0$, $E(X) = \mu$, $Var(X) = \sigma^2$, a standard normal distribution is $Z \sim N(0, 1^2)$, pdf: $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$, $E(Z) = 0$, $Var(Z) = 1$, and for any $X \sim N(\mu, \sigma^2)$, we

can normalize it to $Z = \dfrac{X-\mu}{\sigma} \sim N(0, 1^2)$ which is a standard normal distribution, know how to use the z-table

Poisson distribution: $X \sim P(\lambda)$, $p_n = \dfrac{\lambda^n}{n!} e^{-\lambda}$, $E(X) = Var(X) = \lambda$

Geometric distribution: $X \sim G(p)$, $p_n = p^n(1-p)$, $E(X) = \dfrac{p}{1-p}, Var(X) = \dfrac{p}{(1-p)^2}$