MATH 121 FINAL EXAM

There will be **120** minutes for you to finish the exam, if you ever get stuck, move on and circle back to it. **No calculators**, one problem per sheet, 200 points in total

1. (a) (10 points) What is $\int \tan^2 x \mathrm{d}x$

   (b) (10 points) What is $\int (7x-4)e^{2x}\mathrm{d}x$

   (c) (10 points) Evaluate $\int \frac{8x}{e^{x^2}}\mathrm{d}x$

   (d) (10 points) Evaluate $\int_0^{\pi} x\sin(8x)\mathrm{d}x$

2. (a) (10 points) If $f(t)$ is the solution to the initial value problem $y' = e^t + y, y(0) = 0$, find $f(0), f'(0)$

   (b) (10 points) Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{t^2y^2}{t^3+8}$

   (c) (10 points) Solve the initial value problem $y' + y = e^{2t}, y(0) = -1$

3. (a) (10 points) Evaluate $\sum_{k=0}^{\infty} \frac{3^k}{4^{k+1}}$

   (b) (10 points) Determine which rational number has decimal expansion $2.7\overline{18}$

   (c) (10 points) Determine if the following series converges $\sum_{k=2}^{\infty} \frac{1}{k\sqrt{\ln k}}$

   (d) (10 points) What is the Taylor series of $\frac{2}{x^{-2}e^{x^3}}$ at $x = 0$, include at least four non-zero terms

4. (a) (20 points) Suppose you are playing a game, there are three choices, you can take a bet, if you choose to take bet A, there is 10% probability of winning \$5000, 20% probability of lossing \$2000, hence 70% probability of not lossing or gaining anything, if you choose to take bet B, then there is 28% probability of winning \$1000, 72% probability of lossing \$100, and if you don't bet at all, you are guaranteed to win \$200, should you bet? if so, which bet should you choose?

5. (a) (20 points) The time(in minutes) required to complete an assembly on a production line is a random variable X with cumulative distribution function $F(x) = \frac{1}{125}x^3, 0 \le x \le 5$

   (a) Find $E(X)$

   (B) Compute $Var(X)$

6. (a) (30 points) In a certain town, there are two competing taxicab companies, Red Cab and Blue Cab. The taxis mix with downtown traffic in a random manner. There are five times as many Red taxis as Blue taxis. Suppose you stand on a downtown street and count the number X of Red taxis before the first Blue taxi appears
   (a) Determine the formula for $Pr(X = n)$
   (b) What is the likelihood of observing at least four Red taxis before the first Blue taxi?
   (c) What is the average number of consecutive Red taxis prior to the appearance of a Blue taxi?

7. (a) (20 points) Let X be a normal distribution with expected value 3 and variance 4
   (a) what is the probability of $X \geq 7$
   (a) what is the probability of $X \leq -1$

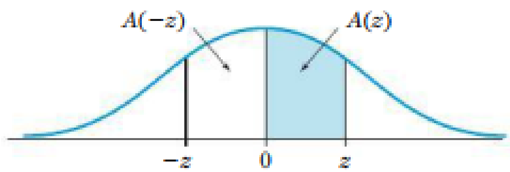

TABLE 1 Areas under the Standard Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0754
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2258	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2518	.2549
0.7	.2580	.2612	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2996	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3820
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998

1. (a) $\int \tan^2 x\,dx = \int \sec^2 x - 1\,dx = \tan x - x + C$

(b) $\int (7x-4)e^{2x}dx \overset{u=2x}{=\!=\!=} \int \left(\frac{7}{2}u - 4\right)e^u \frac{1}{2}du = \int \frac{7}{4}ue^u du - \int 2e^u du$

$= \frac{7}{4}(ue^u - e^u) - 2e^u + C = \frac{7}{4}ue^u - \frac{15}{4}e^u + C = \frac{7}{2}xe^{2x} - \frac{15}{4}e^{2x} + C$

(c) $\int \frac{8x}{e^{x^2}}dx = \int 8xe^{-x^2}dx \overset{u=x^2}{=\!=\!=} 4\int e^{-u}du = -4e^{-u} + C = -4e^{-x^2} + C$

(d) $\int_0^{\pi} x\sin(8x)\,dx \overset{u=8x}{=\!=\!=} \frac{1}{8}\int_0^{8\pi} u\sin u\,du = \frac{1}{8}\left[-u\cos u + \sin u\right]\Big|_0^{8\pi} = -\pi$

2. (a) $f(0) = 0$, $f'(0) = e^0 + f(0) = 1$

(b) $\frac{dy}{dt} = \frac{t^2y^2}{t^3+8} \Rightarrow \frac{dy}{y^2} = \frac{t^2dt}{t^3+8} \Rightarrow -\frac{1}{y} = \frac{1}{3}\ln(t^3+8) + C \Rightarrow y = -\frac{1}{\frac{1}{3}\ln(t^3+8)+C}$

(c) $y' + y = e^{2t} \Rightarrow (e^t y)' = e^{3t} \Rightarrow e^t y = \frac{1}{3}e^{3t} + C$, plug in $y(0) = -1$, $C = -\frac{4}{3}$

thus $y(t) = \frac{1}{3}e^{2t} - \frac{4}{3}e^{-t}$

3. (a) $\sum_{k=0}^{\infty} \frac{3^k}{4^{k+1}} = \frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{3}{4}\right)^k = \frac{1}{4}\cdot\frac{1}{1-\frac{3}{4}} = 1$

(b) $2.7\overline{18} = 2.7 + 0.0\overline{18} = 2.7 + \frac{0.018}{1-0.01} = 2.7 + \frac{2}{110} = \frac{299}{110}$

(c) Let $f(x) = \frac{1}{x\sqrt{\ln x}}$, $x \geq 2$, $f(x)$ is continuous and decreasing, thus we can use the integral test : $\int_2^{\infty} f(x)dx = \lim_{b\to\infty}\int_2^b \frac{1}{x\sqrt{\ln x}}dx \overset{u=\ln x}{=\!=\!=} \lim_{b\to\infty}\int_2^{\ln b} u^{-\frac{1}{2}}du =$

$\lim_{b\to\infty}\left[2u^{\frac{1}{2}}\right]\Big|_2^{\ln b} = \lim_{b\to\infty}\left[2(\ln b)^{\frac{1}{2}} - 2\sqrt{2}\right] = \infty$

Thus $\sum_{k=2}^{\infty}\frac{1}{k\sqrt{\ln k}}$ does not converge

(d) $\frac{2}{x^{-2}e^{x^3}} = 2x^2e^{-x^3} = 2x^2\left(1 + (-x^3) + \frac{1}{2!}(-x^3)^2 + \frac{1}{3!}(-x^3)^3 + \cdots\right)$

$= 2x^2\left(1 - x^3 + \frac{1}{2}x^6 - \frac{1}{6}x^9 + \cdots\right) = 2x^2 - 2x^5 + x^8 - \frac{1}{3}x^{11} + \cdots$

4. (a) Let X be the money you gain

Choose bet A : $E(X) = 5000\cdot 0.1 + (-2000)\cdot 0.2 + 0\cdot 0.7 = 100$

Choose bet B : $E(X) = 1000\cdot 0.28 + (-100)\cdot 0.72 = 208$

Don't bet : $E(X) = 200\cdot 1 = 200$

But then $208 > 200 > 100$

Therefore you should bet for B

5. The probability density function is $f(x) = F'(x) = \frac{3}{125}x^2$

(a) $E(X) = \int_0^5 x f(x)\,dx = \int_0^5 \frac{3}{125}x^3\,dx = \frac{3}{500}x^4\Big|_0^5 = \frac{15}{4}$

(b) $Var(X) = \int_0^5 x^2 f(x)\,dx - E(X)^2 = \int_0^5 \frac{3}{125}x^4\,dx - \left(\frac{75}{4}\right)^2 = \frac{3}{625}x^5\Big|_0^5 - \left(\frac{15}{4}\right)^2 = \frac{15}{16}$

6. X satisfies geometric distribution with $p = \frac{5}{6}$, where p is the probability of a Red taxi showing up

(a) $Pr(X=n) = \left(\frac{5}{6}\right)^n\left(\frac{1}{6}\right)$

(b) $Pr(X \geq 4) = 1 - Pr(X \leq 3) = 1 - P_0 - P_1 - P_2 - P_3 = 1 - \frac{1}{6} - \frac{5}{6}\frac{1}{6} - \left(\frac{5}{6}\right)^2\frac{1}{6} - \left(\frac{5}{6}\right)^3\frac{1}{6} = \left(\frac{5}{6}\right)^4 = \frac{625}{1296}$

(c) $E(X) = \frac{p}{1-p} = \frac{\frac{5}{6}}{1-\frac{5}{6}} = 5$

7. Since X satisfies a normal distribution with expected value 3 and variance 4, $\frac{X-3}{2} =: Z$ satisfies standard normal distribution

(a) $Pr(X \geq 7) = Pr\left(\frac{X-3}{2} \geq \frac{7-3}{2}\right) = Pr(Z \geq 2) = Pr(Z \geq 0) - Pr(0 \leq Z \leq 2)$

$= \frac{1}{2} - A(2) = 0.5 - 0.4772 = 0.228$

(b) $Pr(X \leq -1) = Pr\left(\frac{X-3}{2} \leq \frac{-1-3}{2}\right) = Pr(Z \leq -2) = Pr(Z \geq 2) = 0.228$