- You should have **SIX** answer sheets. Use exactly ONE answer sheet per question. Put your name, your TA's name, your section number and question number on EACH answer sheet.
- No books, notebooks, calculators, cell phones or other electronic devices.
- **YOU MUST SHOW ALL APPROPRIATE WORK IN ORDER TO RECEIVE FULL CREDIT FOR AN ANSWER.** Show enough work that we can follow your thinking.
- Before handing in your test: on your *first* answer sheet only, please copy the pledge and sign.

ANSWERS SHOULD BE EXACT AND IN SIMPLEST FORM UNLESS OTHERWISE INDICATED.

Answer question 1 on answer page 1. Use both sides if necessary.

1.a. (10 points) Let $f(x)=\frac{1}{2}x-\sin x$. Find the value $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$ for which *f* has a horizontal tangent.

1.b. (12 points) Find the slope of the curve $y=x^2\sin(2x)$ at $x=\frac{\pi}{2}$.

1.c. (12 points) Find the area under the curve $y=x\sin\left(x^2\right)$ between $x=0$ and $x=\frac{\pi}{2}$.

Answer question 2 on answer page 2. Use both sides if necessary.

2.a. (10 points) Use the Trapezoidal Rule with $n=3$ partitions to approximate the area under the curve $y=x^3$ on the interval $1\le x\le 3$.

2.b. (12 points) Evaluate $\int x^3\ \ln x\, dx$.

2.c. (12 points) Evaluate $\int\frac{2x}{\left(1+x^2\right)^3}\,dx$.

Answer question 3 on answer page 3. Use both sides if necessary.

3.a. (10 points) Find all constant solutions for the differential equation $y'-y^2=2y-3$.

3.b. (14 points) Find the general solution of the differential equation $y'-\frac{2}{x}y+1=0$.

3.c. (14 points) Find the general solution of the differential equation $y'=\frac{x^3}{y^4}$.

Answer question 4 on answer page 4. Use both sides if necessary.

4.a. (10 points) Medication is introduced via an intravenous drip at a rate of 0.5 mg per hour. On a continuous basis 2% of the drug in the bloodstream is absorbed into the body. Determine the equilibrium amount of the medication in the bloodstream of the patient. That is, after a lengthy period of time, how much medication would we expect to be in the patient's bloodstream?

4.b. (14 points) Use derivatives to find the third Taylor polynomial, $p_3(x)$, for $y=3e^{-2x}$, centered about $x=0$.

Answer question 5 on answer page 5. Use both sides if necessary.

5.a. (10 points) Determine whether the series $\sum_{k=1}^{\infty}ke^{-3k}$ converges or diverges.

5.b. (10 points) Find the sum of the series $\sum_{k=2}^{\infty}\frac{3}{4^k}$.

5.c. (10 points) Using suitable operations on a known Taylor series, derive a series expansion for $p(x)=xe^{-x^3}$. Include at least four non-zero terms.

CONTINUED ON BACK

Answer question 6 on answer page 6. Use both sides if necessary.

6.a. (15 points) In a factory, the random variable X = " number of years for a machine to work before needing to be replaced" has probability density function $f(t) = 0.4e^{-0.4t}$, $0 \le t \le \infty$. First, verify that $f(t)$ meets both criteria for being a probability density function. Then, calculate the probability that a particular machine will last between 2 and 5 years from the time it was new.

6.b. (15 points) A continuous random variable X has cumulative distribution function $F(x) = \frac{1}{7}x^3 - \frac{1}{7}$, $1 \le x \le 2$. Find $E(X)$ and Var(X). [Hint: First find the probability density function $f(x)$].

6.c. (10 points) The length of a spotted salamander measured (then released) in a northwest U.S. national park is a normal random variable with mean μ = 8 inches and standard deviation σ = 2. What is the probability that a randomly chosen salamander is less than 5 inches long?

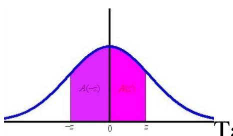

Table 1 Areas under the Standard Normal Curve

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857

1. (a). $f'(x)=\frac{1}{2}-\cos x$, $f'(x)=0 \Leftrightarrow \cos x=\frac{1}{2}$, also $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$, thus $x=\frac{\pi}{3}, -\frac{\pi}{3}$

(b) $y'(x)=2x\sin 2x+2x^2\cos 2x$, $y'(\frac{\pi}{2})=2\cdot\frac{\pi}{2}\sin\pi+2(\frac{\pi}{2})^2\cos\pi=-\frac{\pi^2}{2}$

(c) $\int_0^{\frac{\pi}{2}} x\sin x^2 dx \overset{u=x^2}{=\!=\!=} \int_0^{\frac{\pi^2}{4}}\frac{1}{2}\sin u\,du=\frac{1}{2}\int_0^{\frac{\pi^2}{4}}\sin u\,du=\frac{1}{2}(-\cos u)\Big|_0^{\frac{\pi^2}{4}}$

$=\frac{1}{2}\left[\left(-\cos\frac{\pi^2}{4}\right)-\left(-\cos 0\right)\right]=\frac{1}{2}\left[1-\cos\left(\frac{\pi^2}{4}\right)\right]$

2. (a) $n=3$, $\Delta x=\frac{3-1}{3}=\frac{2}{3}$,

$\int_1^3 x^3dx \approx \frac{\frac{2}{3}}{2}\left[1^3+2\cdot\left(\frac{5}{3}\right)^3+2\cdot\left(\frac{7}{3}\right)^3+3^3\right]=\frac{1}{3}\left[1+2\cdot\frac{125}{27}+2\cdot\frac{343}{27}+27\right]$

$=\frac{1}{3}\left[1+\frac{250}{27}+\frac{686}{27}+27\right]=\frac{1}{3}\cdot\frac{1692}{27}=\frac{188}{9}$

(b) $\int x^3\ln x\,dx=\frac{1}{4}\int(x^4)'\ln x\,dx=\frac{1}{4}\left[x^4\ln x-\int(\ln x)'x^4dx\right]$

$=\frac{1}{4}x^4\ln x-\frac{1}{4}\int\frac{1}{x}x^4dx=\frac{1}{4}x^4\ln x-\frac{1}{4}\int x^3dx=\frac{1}{4}x^4\ln x-\frac{x^4}{16}+C$

(c) $\int\frac{2x}{(1+x^2)^3}dx \underset{du=2xdx}{\overset{u=1+x^2}{=\!=\!=}} \int\frac{1}{u^3}du=\int u^{-3}du=-\frac{u^{-2}}{2}+C=-\frac{(1+x^2)^{-2}}{2}+C$

3. (a) $y'=y^2+2y-3=(y+3)(y-1)$, if y is a constant solution, $y'=0 \Rightarrow$
$0=(y+3)(y-1)\Rightarrow y=-3$ or $y=1$

(b) $y'+(-\frac{2}{x})y=-1$ $\qquad a(x)=-\frac{2}{x}$, $b(x)=-1$

$A(x)=-2\ln x=\ln x^{-2}$, $e^{A(x)}=e^{\ln x^{-2}}=x^{-2}$

$[yx^{-2}]'=-x^{-2}\Rightarrow yx^{-2}=\int -x^{-2}dx=x^{-1}+C$

$\Rightarrow y=x+Cx^2$

(c) $y'=\frac{x^3}{y^4}\Rightarrow\frac{dy}{dx}=\frac{x^3}{y^4}\Rightarrow y^4dy=x^3dx\Rightarrow\int y^4dy=\int x^3dx$

$\Rightarrow\frac{y^5}{5}=\frac{x^4}{4}+C\Rightarrow y^5=\frac{5}{4}x^4+5C\Rightarrow y=\sqrt{\frac{5}{4}x^4+5C}$

4. (a) $0.5\cdot 0.98+0.5\cdot 0.98^2+0.5\cdot 0.98^3+\cdots=\frac{0.5\cdot 0.98}{1-0.98}=\frac{0.49}{0.02}=\frac{49}{2}$

(b) $p_3(x)=3-6x+6x^2-4x^3$

5. (a) $f(x)=xe^{-3x}$ is positive, continuous and decreasing

$$\int_1^{\infty} xe^{-3x}dx = \lim_{b\to+\infty}\int_1^b xe^{-3x}dx = \lim_{b\to+\infty} -\frac{1}{3}\int_1^b x(e^{-3x})'dx$$

$$= \lim_{b\to+\infty} -\frac{1}{3}\left[xe^{-3x}\Big|_1^b - \int_1^b e^{-3x}dx\right] = \lim_{b\to+\infty} -\frac{1}{3}\left[be^{-3b} - e^{-3} + \frac{1}{3}e^{-3x}\Big|_1^b\right]$$

$$= \lim_{b\to+\infty} -\frac{1}{3}\left[be^{-3b} - e^{-1} + \frac{1}{3}(e^{-3b} - e^{-3})\right] = -\frac{1}{3}\left(0 - e^{-1} + \frac{1}{3}(0 - e^{-3})\right) \text{ which is finite}$$

thus the series converges

(b) $$\sum_{k=2}^{\infty}\frac{3}{4^k} = \frac{\frac{3}{16}}{1-\frac{1}{4}} = \frac{\frac{3}{16}}{\frac{3}{4}} = \frac{1}{4}$$

(c) $$xe^{-x^3} = x\left[1 + (-x^3) + \frac{(-x^3)^2}{2!} + \frac{(-x^3)^3}{3!} + \cdots\right]$$

$$= x\left[1 - x^3 + \frac{x^6}{2} - \frac{x^9}{6} + \cdots\right] = x - x^4 + \frac{x^7}{2} - \frac{x^{10}}{6} + \cdots$$

6. (a) $f(t) \geq 0$, $0 \leq t \leq \infty$

$$\int_0^{\infty} f(t)dt = \int_0^{\infty} 0.4e^{-0.4t}dt = \lim_{b\to+\infty}\int_0^b 0.4e^{-0.4t}dt \overset{u=0.4t}{=\!=\!=} \lim_{b\to+\infty}\int_0^{0.4b} e^{-u}du$$

$$= \lim_{b\to\infty}(-e^{-u})\Big|_0^{0.4b} = \lim_{b\to\infty}\left[-e^{-0.4b} - (-1)\right] = (-0-(-1)) = 1$$

Thus $f(t)$ is a probability density function

$$Pr(2 \leq X \leq 5) = \int_2^5 f(t)dt = \int_2^5 0.4e^{-0.4t}dt \overset{u=0.4t}{=\!=\!=} \int_{0.4\cdot 2}^{0.4\cdot 5} e^{-u}du = \int_{0.8}^2 e^{-u}du$$

$$= -e^{-u}\Big|_{0.8}^2 = (-e^{-2}) - (-e^{-0.8}) = e^{-0.8} - e^{-2}$$

(b) $$f(x) = F'(x) = \frac{3}{7}x^2,\quad E(X) = \int_1^2 xf(x)dx = \int_1^2 \frac{3}{7}x^3dx = \frac{3}{28}x^4\Big|_1^2 = \frac{3}{28}(2^4 - 1^4) = \frac{3}{28}\cdot 15 = \frac{45}{28}$$

$$Var(X) = \int_1^2 x^2f(x)dx - E(X)^2 = \int_1^2 \frac{3}{7}x^4dx - \left(\frac{45}{28}\right)^2 = \frac{3}{35}x^5\Big|_1^2 - \left(\frac{45}{28}\right)^2$$

$$= \frac{3}{35}(2^5 - 1^5) - \left(\frac{45}{28}\right)^2 = \frac{93}{35} - \frac{2025}{784} = \frac{291}{3920}$$

(c) $X \sim N(\mu, \sigma^2) = N(8, 2^2)$, then $Z = \frac{X-8}{2} \sim N(0, 1^2)$

$$Pr(X \leq 5) = Pr\left(\frac{X-8}{2} \leq \frac{5-8}{2}\right) = Pr\left(Z \leq -\frac{3}{2}\right) = Pr(Z \leq 0) - Pr\left(-\frac{3}{2} \leq Z \leq 0\right)$$

$$= \frac{1}{2} - Pr\left(0 \leq Z \leq \frac{3}{2}\right) = \frac{1}{2} - A(1.5) = 0.5 - 0.4332 = 0.0668$$