# MATH 121 EXAM 3

| |
|---|
| There will be **50** minutes for you to finish the exam, if you ever get stuck, move on and circle back to it. **No calculators**, one problem per sheet, 100 points in total |

1. Determine if the following series or improper integrals are convergent

   (a) (20 points) $\displaystyle\sum_{k=0}^{\infty} \frac{k+2}{k^2+2k+1}$

   **Solution:** First notice $\displaystyle\sum_{k=0}^{\infty} \frac{k+2}{k^2+2k+1} = \sum_{k=0}^{\infty} \frac{k+2}{(k+1)^2}$, let $f(x) = \dfrac{x+2}{(x+1)^2}$ which is continuous, positive and decreasing function from 1 to $\infty$, we can apply the integral test

$$\begin{aligned}
\int_0^{\infty} f(x)dx &= \int_1^{\infty} \frac{x+2}{(x+1)^2}dx = \lim_{b\to\infty} \int_0^{b} \frac{x+2}{(x+1)^2}dx \qquad \text{let } u = x+1, \text{then } x = u-1, \text{ and } du = dx \\
&= \int_{0+1}^{b+1} \frac{u+1}{u^2}du = \int_1^{b+1} \frac{u+1}{u^2}du = \int_1^{b+1} \left(\frac{u}{u^2} + \frac{1}{u^2}\right) du = \int_1^{b+1} \left(u^{-1} + u^{-2}\right) du \\
&= \int_1^{b+1} u^{-1}du + \int_1^{b+1} u^{-2}du = \ln u|_2^{b+1} + \left.\frac{u^{-1}}{-1}\right|_2^{b+1} = \ln u|_1^{b+1} + \left.\left(-\frac{1}{u}\right)\right|_1^{b+1} \\
&= (\ln(b+1) - \ln 1) + \left(\left(-\frac{1}{b+1}\right) - \left(-\frac{1}{1}\right)\right)
\end{aligned}$$

   As $b \to \infty$, $\ln(b+1) \to \infty$, $\dfrac{1}{b+1} \to 0$, thus the improper integral is not convergent, and the series is also not convergent

   (b) (10 points) $\displaystyle\int_1^{\infty} \frac{5^x}{7^x}dx$

   **Solution:** Let $f(x) = \dfrac{5^x}{7^x}$, which is continuous, positive and decreasing function from 1 to $\infty$, so we can use the integral test

   But $\displaystyle\sum_{k=1}^{\infty} \frac{5^k}{7^k}$ is a geometric series with $a = \frac{5}{7}$, $r = \frac{5}{7} < 1$, thus $\displaystyle\sum_{k=1}^{\infty} \frac{5^k}{7^k}$ is convergent and $\displaystyle\int_1^{\infty} \frac{5^x}{7^x}dx$ is convergent

2. (a) (10 points) Evaluate $\displaystyle\sum_{k=0}^{\infty} \frac{1}{2^{3k}}$

   **Solution:** $\displaystyle\sum_{k=0}^{\infty} \frac{1}{2^{3k}} = \sum_{k=0}^{\infty} \frac{1}{(2^3)^k} = \sum_{k=0}^{\infty} \frac{1}{8^k} = 1 + \frac{1}{8} + \frac{1}{8^2} + \frac{1}{8^3} + \cdots = \frac{1}{1-\frac{1}{8}} = \frac{1}{\frac{7}{8}} = \frac{8}{7}$

   (b) (10 points) Determine which rational number has decimal expansion $9.\overline{99}$

   **Solution:** First notice that $9.\overline{99} = 9.\overline{9} = 9 + 0.9 + 0.09 + 0.009 + 0.0009 + \cdots = \frac{9}{1-0.1} = \frac{9}{0.9} = 10$

   So miraculously we have $9.\overline{9} = 9.99999\cdots = 10$

   (c) (10 points) Suppose $f(x) = 5(x-1)^3 - 17(x-1)^{11}$, compute $f^{(11)}(1)$

**Solution:** We know that the Taylor expansion of $f(x)$ at $x = 1$ is $f(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \cdots + \frac{f^{(11)}(1)}{11!}(x-1)^{11} + \cdots$, thus by comparing the coefficients we have $-17 = \frac{f^{(11)}(1)}{11!} \Rightarrow f^{(11)}(1) = -17 \cdot 11!$

3. (a) (20 points) Evaluate $1 + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \frac{6}{32} + \frac{7}{64} + \cdots$

**Hint:** what is the derivative of $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$

**Solution:** Take the derivative of the function $\frac{1}{1-x}$ on the left hand side and series $\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \cdots$ on the right hand side, we have
$\frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)' = \left(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \cdots\right)' = 1' + x' + (x^2)' + (x^3)' + (x^4)' + (x^5)' + (x^6)' + (x^7)' + \cdots = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + \cdots$
notice that if we plug in $x = \frac{1}{2}$, we have $4 = \frac{1}{\left(1-\frac{1}{2}\right)^2} = 1 + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \frac{6}{32} + \frac{7}{64} + \cdots$

4. (a) (20 points) A cement company plans to bid on a contract for constructing the foundations of new homes in a housing development. The company is considering two bids: a high bid that will produce $\$75,000$ profit(if the bid is accepted) and a low bid that will produce $\$40,000$ profit.From past experience, the company estimates that the high bid has a 30% chance of acceptance and the low bid a 50% chance. Which bid should the company make?

**Solution:** This is exactly the same problem from the textbook!

Let $X$ denote the profit

If the company make the high bid, then the probability table would be

| | | |
|---|---|---|
| $X$ | 75000 | 0 |
| $P$ | 0.3 | 0.7 |

Then the expectation will be $E(X) = 75000 \cdot 0.3 + 0 \cdot 0.7 = 22500$

If the company make the low bid, then the probability table would be

| | | |
|---|---|---|
| $X$ | 40000 | 0 |
| $P$ | 0.5 | 0.5 |

Then the expectation will be $E(X) = 40000 \cdot 0.5 + 0 \cdot 0.5 = 20000$

Since $22500 > 20000$, the company should make the high bid