MATH 121 EXAM 2

There will be **50** minutes for you to finish the exam, if you ever get stuck, move on and circle back to it. **No calculators**, one problem per sheet, 100 points in total

1. Determine if the following improper integrals are convergent, if it is, evaluate it

   (a) (10 points) $\int_2^{+\infty} \frac{1}{x\ln x}\mathrm{d}x$

   Solution: $\int_2^{+\infty} \frac{1}{x\ln x}\mathrm{d}x = \lim_{b\to+\infty}\int_2^{b} \frac{1}{x\ln x}\mathrm{d}x \overset{u=\ln x, du=\frac{1}{x}dx}{=\!=\!=\!=} \lim_{b\to+\infty}\int_{\ln 2}^{\ln b}\frac{1}{u}du = \lim_{b\to+\infty} \ln u\Big|_{\ln 2}^{\ln b} = \lim_{b\to+\infty}\left[\ln(\ln b) - \ln(\ln 2)\right] = +\infty$, thus the improper integral is not convergent

   (b) (10 points) $\int_e^{+\infty} \frac{1}{x\ln^2 x}\mathrm{d}x = \int_e^{+\infty} \frac{1}{x(\ln x)^2}\mathrm{d}x$

   Solution: $\int_e^{+\infty} \frac{1}{x\ln^2 x}\mathrm{d}x = \lim_{b\to+\infty}\int_e^{b} \frac{1}{x\ln^2 x}\mathrm{d}x = \lim_{b\to+\infty} -\left[\frac{1}{\ln b} - \frac{1}{\ln e}\right] = 1$

2. Solve the following differential equation with initial condition

   (a) (20 points) $y' = \sqrt[3]{\frac{27t^2}{8y}}, y(0) = -8$

   Solution: Use separation of variables

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} = \sqrt[3]{\frac{27t^2}{8y}} = \frac{3t^{\frac{2}{3}}}{2y^{\frac{1}{3}}} &\Rightarrow 2y^{\frac{1}{3}}\mathrm{d}y = 3t^{\frac{2}{3}}\mathrm{d}t \\ &\Rightarrow \int 2y^{\frac{1}{3}}\mathrm{d}y = \int 3t^{\frac{2}{3}}\mathrm{d}t \\ &\Rightarrow \frac{3}{2}y^{\frac{4}{3}} = \frac{9}{5}t^{\frac{5}{3}} + C\end{aligned}$$

   Plug in initial condition $y(0) = -8$, we have $\frac{3}{2}\times(-8)^{\frac{4}{3}} = \frac{9}{5}\times 0^{\frac{5}{3}} + C \Rightarrow C = 24$, hence $\frac{3}{2}y^{\frac{4}{3}} = \frac{9}{5}t^{\frac{5}{3}} + 24 \Rightarrow y^{\frac{4}{3}} = \frac{6}{5}t^{\frac{5}{3}} + 16 \Rightarrow y = -\left(\frac{6}{5}t^{\frac{5}{3}} + 16\right)^{\frac{3}{4}} = -\sqrt[4]{\left(\frac{6}{5}t^{\frac{5}{3}} + 16\right)^3}$, solution $\sqrt[4]{\left(\frac{6}{5}t^{\frac{5}{3}} + 16\right)^3}$ is discarded since it doesn't satisfy the initial condition

3. A person deposits an inheritance of \$50000 in a savings account that earns 4% interest compounded continuously. This person intends to make withdrawals that will increase gradually in size with time. Suppose that the annual rate of withdrawals is $2000 + 500t$, dollars per year, t years from the time the account was opened.

(a) (5 points) Assume the withdrawals are made at a continuous rate. Set up a differential equation with initial condition that is satisfied by the amount $f(t)$(dollars) in the account at time t(years)

Solution: $y' = 0.04y - (2000 + 500t), y(0) = 50000$

(b) (25 points) Determine $f(t)$

Solution: The integrating factor is $e^{-0.04t}$, then we have $ye^{-0.04t} = -\int (2000 + 500t)e^{-0.04t}dt = (12500t + 362500)e^{-0.04t} + C$, plug in the initial condition $y(0) = 50000$, we have $C = -312500$, thus $y = 12500t + 362500 - 312500e^{0.04t}$

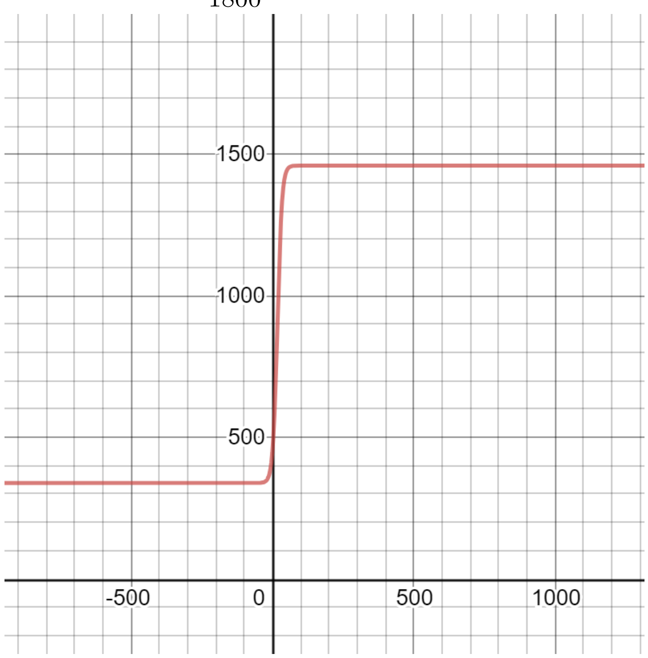
4. The fish population in a pond with carrying capacity of 1800 fish is modeled by the logistic equation, $\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{0.2}{1800}N\,(1800 - N)$. Here N denotes the number of fish at time t in years, the owner of the pond decided to put in 450 fish in the beginning and remove 55 fish per year

(a) (10 points) Modify the differential equation to model the fish population

Solution: $\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{0.2}{1800}N\,(1800 - N) - 55$

(b) (20 points) Plot the solution curve of the new equation with $N(0) = 450$, is the practice of catching 55 fish per year sustainable, or will it deplete the fish population in the pond?

Solution: Since $\frac{0.2}{1800} \times 450 \times (1800 - 450) - 55 = 67.5 - 55 = 12.5 > 0$, this practice is sustainable



(c) (5 points (bonus)) If the owner wants to maximize his profits by changing his practice, namely by changing the number of fish putting into the empty pond in the beginning and the number of fish removing from the pond per year, what is the maximal number of fish that he can remove per year such that the practice is still sustainable?(Only answer is needed)

Solution: There answer is $90 = \frac{0.2}{1800} \times 900 \times (1800 - 900)$, the graph of $z = \frac{0.2}{1800}N\,(1800 - N) - K$ still intersects the N axis, where K is the number of fish been removed per year