# Arithmetic and Formulas

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## 1 Notations

$\pm$ means $+$ or $-$, $\mp$ means $-$ or $+$, for example $1 \pm \sqrt{2}$ means $1+\sqrt{2}$ or $1-\sqrt{2}$
Composition: $f \circ g(x) := f(g(x))$, for example if $f(x) = \sqrt{x}, g(x) = e^x$, then $f(g(x)) = \sqrt{g(x)} = \sqrt{e^x}$, if $f(x) = \frac{1}{x}, g(x) = \sin x$, then $f(g(x)) = \frac{1}{g(x)} = \frac{1}{\sin x}$

## 2 Fractions

Scaling: $\frac{a}{b} = \frac{ac}{bc}, c \neq 0$, for example $\frac{2}{3} = \frac{2\cdot 5}{3\cdot 5} = \frac{10}{15}$
Summation(Subtraction): $\frac{a}{b} \pm \frac{c}{d} = \frac{ad}{bd} \pm \frac{bc}{db} = \frac{ad \pm bc}{bd}$, for example $\frac{7}{5} + \frac{2}{3} = \frac{7\cdot 3}{5\cdot 3} + \frac{2\cdot 5}{3\cdot 5} = \frac{7\cdot 3 + 2\cdot 5}{5\cdot 3} = \frac{21+10}{15} = \frac{31}{15}$, $\frac{2}{5} - 2 = \frac{2}{5} - \frac{2}{1} = \frac{2}{5} - \frac{2\cdot 5}{1\cdot 5} = \frac{2}{5} - \frac{10}{5} = \frac{2-10}{5} = \frac{-8}{5} = -\frac{8}{5}$
Multiplication: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$, for example $\frac{7}{5} \cdot \frac{2}{3} = \frac{7\cdot 2}{5\cdot 3} = \frac{14}{15}$, $\frac{4}{7} \cdot 3 = \frac{4}{7} \cdot \frac{3}{1} = \frac{4\cdot 3}{7\cdot 1} = \frac{12}{7}$
Division: $\frac{a}{b} \Big/ \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$, for example $\frac{7}{5} \Big/ \frac{2}{3} \frac{7}{5} \cdot \frac{3}{2} = \frac{7\cdot 3}{5\cdot 2} = \frac{21}{10}$, $\frac{4}{7} \Big/ 3 = \frac{4}{7} \Big/ \frac{3}{1} = \frac{4}{7} \cdot \frac{1}{3} = \frac{4\cdot 1}{7\cdot 3} = \frac{4}{21}$

## 3 Exponential

$A^0 = 1, A \neq 0$
$A^{-a} = \frac{1}{A^a}$, for example $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$
$A^{\frac{1}{a}} = \sqrt[a]{A}$ for example $\sqrt{3} = 3^{\frac{1}{2}}$, $5^{\frac{1}{3}} = \sqrt[3]{5}$
$A^a \cdot A^b = A^{a+b}$, for example $2^3 \cdot 2^{-1} = 2^{3+(-1)} = 2^{3-1} = 2^2 = 4$
$(A^a)^b = A^{ab}$, for example $\left(3^2\right)^3 = 3^{2\cdot 3} = 3^6 = 729$
$(AB)^a = A^a \cdot B^a$, for example $(3\cdot 5)^{\frac{1}{2}} = 3^{\frac{1}{2}} \cdot 5^{\frac{1}{2}}$

## 4 Logarithms

$\log_e(x)$ is often denoted as $\ln(x)$ or simply $\ln x$, so remember $\ln x$ is a function of $x$
$\log_a(x) = y \Leftrightarrow a^y = x$, for example $\log_e 1 = \ln 1 = 0$ because $e^0 = 1$
$\log_a(xy) = \log_a(x) + \log_a(b)$, $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$, for example $\ln(2\cdot 3) = \ln 2 + \ln 3$, $\ln\left(\frac{5}{7}\right) = \ln 5 - \ln 7$
$\log_a\left(x^b\right) = b\log_a x$, for example $\ln\sqrt{x} = \ln\left(x^{\frac{1}{2}}\right) = \frac{1}{2}\ln x$

## 5 Derivatives

Summation(Subtraction): $(af(x) \pm bg(x))' = af'(x) \pm bg'(x)$, derivative is **distributive**, for example $(e^x - 2\sin x)' = (1\cdot e^x + (-2)\cdot \sin x)' = 1\cdot (e^x)' + (-2)(\sin x)' = (e^x)' - 2(\sin x)'$, or simply $(e^x - 2\sin x)' = (e^x)' - 2(\sin x)' = e^x + 2\cos x$
Product: $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$, for example $(e^x \sin x)' = (e^x)'\sin x + e^x(\sin x)' = e^x \sin x + e^x \cos x$
Composition(Chain rule): $[f(g(x))]' = f'(g(x))g'(x)$, for example $\left[\ln(1+5x^2)\right]'$, let $f(x) = \ln x, g(x) = 1+5x^2$, then $f'(x) = \frac{1}{x}, g'(x) = 10x$, $\left[\ln(1+5x^2)\right]' = [f(g(x))]' = f'(g(x))g'(x) = \frac{1}{g(x)}g'(x) = \frac{1}{1+5x^2} \cdot 10x = \frac{10x}{1+5x^2}$

Quotient: $\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$, notice quotient is a product with composition $\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}$

$(x^n)' = nx^{n-1}$, $(e^x)' = e^x$, $(\ln x)' = \frac{1}{x} = x^{-1}$,$(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $(\tan x)' = \sec^2 x$

# 6 Differentials

Summation(Subtraction): $d\,(af \pm bg) = adf \pm bdg$, where $a, b$ are constants, in other words, $d$ is **distributive**
Product: $d\,(fg) = gdf + fdg$
Composition(Chain rule): $df = \frac{df}{dg}dg = f'(g)\,dg$
Quotient: $d\left(\frac{f}{g}\right) = \frac{gdf - fdg}{g^2}$
$\frac{df}{dx} = f'(x) \Rightarrow df = f'(x)dx \Rightarrow \int df = \int f'(x)dx = f(x)$
$dc = c'dx = 0 \Rightarrow d(af + c) = adf$, where $a, c$ are constants
$dx^n = nx^{n-1}dx$, $de^x = e^x dx$, $d\ln x = \frac{1}{x}dx = \frac{dx}{x}$,$d\sin x = \cos x dx$, $d\cos x = -\sin x dx$, $d\tan x = \sec^2 x dx$

# 7 Integrations

$\int (af(x) \pm bg(x))\,dx = a\int f(x)dx \pm b\int g(x)dx$, $\int$ is **distributive**, for example $\int (e^x + 2\cos x)\,dx = \int e^x dx + 2\int \cos x dx = e^x + 2\sin x + C$. Note that adding $C$ is to cover all antiderivatives
$\int f(g(x))\,g'(x)dx = \int f(g(x))\,dg(x)$, this is integration by substitution
$\int f(x)dg(x) = f(x)g(x) - \int g(x)df(x)$, this is integration by parts

# 8 Trigonometry identities

$\cos^2 x + \sin^2 x = 1$
$1 + \tan^2 x = \sec^2 x$
$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$
$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$
$\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}$
$\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$
$\sin(-x) = -\sin x$, $\cos(-x) = \cos x$, $\tan(-x) = -\tan x$, $\tan x = \frac{\sin x}{\cos x}$, $\sec x = \frac{1}{\cos x}$,

# 9 Trigonometry table

| $\theta$ | 0 | $\frac{\pi}{6} = 30°$ | $\frac{\pi}{4} = 45°$ | $\frac{\pi}{3} = 60°$ | $\frac{\pi}{2} = 90°$ |
|---|---|---|---|---|---|
| $\sin\theta$ | 0 | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 |
| $\cos\theta$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 |
| $\tan\theta$ | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | $\times$ |

## 10 Approximation to definite integrals

Given $a = a_0, b = a_n$, and divide interval $(a,b)$ into $n$ equal parts $(a_0,a_1),(a_1,a_2),\cdots,(a_{n-2},a_{n-1}),(a_n,a_n)$ with $\Delta x = a_i - a_{i-1}$ being the length of each subinterval, and let $x_i = \frac{a_{i-1}+a_i}{2}$ be the midpoint of $a_{i-1}$ and $a_i$

Midpoint rule: $\int_a^b f(x)dx \approx [f(x_1)+f(x_2)+\cdots+f(x_{n-1})+f(x_n)]\Delta x$

Trapezoidal rule: $\int_a^b f(x)dx \approx [f(a_0)+2f(a_1)+2f(a_2)+\cdots+2f(a_{n-1})+f(a_n)]\frac{\Delta x}{2}$

Simpson's rule: $\int_a^b f(x)dx \approx [f(a_0)+4f(x_1)+2f(a_1)+4f(x_2)+2f(a_2)+\cdots+2f(a_{n-1})+4f(x_n)+f(a_n)]\frac{\Delta x}{6}$

## 11 Method of Integrating factors

Always starts with a DE(differential equation) of the form $y'+a(t)y=b(t)$. Note that if your DE is $t^2y'+y=t^3$, then first you need to divide $t^2$ on both sides, then you would have $y'+\frac{1}{t^2}y=t$ with $a(t)=\frac{1}{t^2}, b(t)=t$

Then take any antiderivative $A(t)=\int a(t)dt$, such that $A'(t)=a(t)$

Then we get the integrating factor $e^{A(t)}$, for example if $a(t)=2$, then $A(t)$ could be $2t$, then $e^{A(t)}=e^{2t}$

Then we have $\left[ye^{A(t)}\right]'=b(t)e^{A(t)}$

Then we integrate on both sides to get $ye^{A(t)}=\int ye^{A(t)}dt=\int b(t)e^{A(t)}dt$

At last, we divide $e^{A(t)}$ on both sides(which is equivalent to multiplying $e^{-A(t)}$ on both sides), we get $y = e^{-A(t)}\int b(t)e^{A(t)}dt$

For example, suppose the DE we have is $y'+2y=t$

First we identify $a(t)=2, b(t)=t$

Find one antiderivative of $a(t)$, $A(t)=\int 2dt = 2t$, notice that the reason for omitting $C$ is that we only need to find one such antiderivative

The integrating factor is $e^{A(t)}=e^{2t}$

We have $\left[ye^{2t}\right]'=te^{2t}$

We have $ye^{2t}=\int te^{2t}dt=\frac{1}{2}te^{2t}-\frac{1}{4}e^{2t}+C$

Dividing $e^{2t}$(or equivalently, multiplying $e^{-2t}$), we have $y=\frac{1}{2}t-\frac{1}{4}+Ce^{-2t}$

## 12 Taylor polynomials