习题集

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目录

1 数列

习题1.1. 数列$\{a_n\}$满足
$$a_1 = 2, \quad a_{n+1} = 2a_n + 2^{n+1}$$
求$\{a_n\}$的通项公式

解.
$$\begin{aligned} a_{n+1} &= 2a_n + 2^{n+1} \\ \Rightarrow \frac{a_{n+1}}{2^{n+1}} &= \frac{a_n}{2^n} + 1 \\ \Rightarrow a_n &= n2^n \end{aligned}$$

□

习题1.2. 数列$\{a_n\}$满足
$$a_1 = \alpha, \quad a_2 = \beta, \quad a_{n+2} = a_{n+1} - a_n$$
求$\{a_n\}$的通项公式

解.
$$\begin{gathered} a_3 = \beta - \alpha, \quad a_4 = -\alpha \\ a_5 = -\beta, \quad a_6 = \alpha - \beta \\ a_7 = \alpha, \quad a_8 = \beta \end{gathered}$$

□

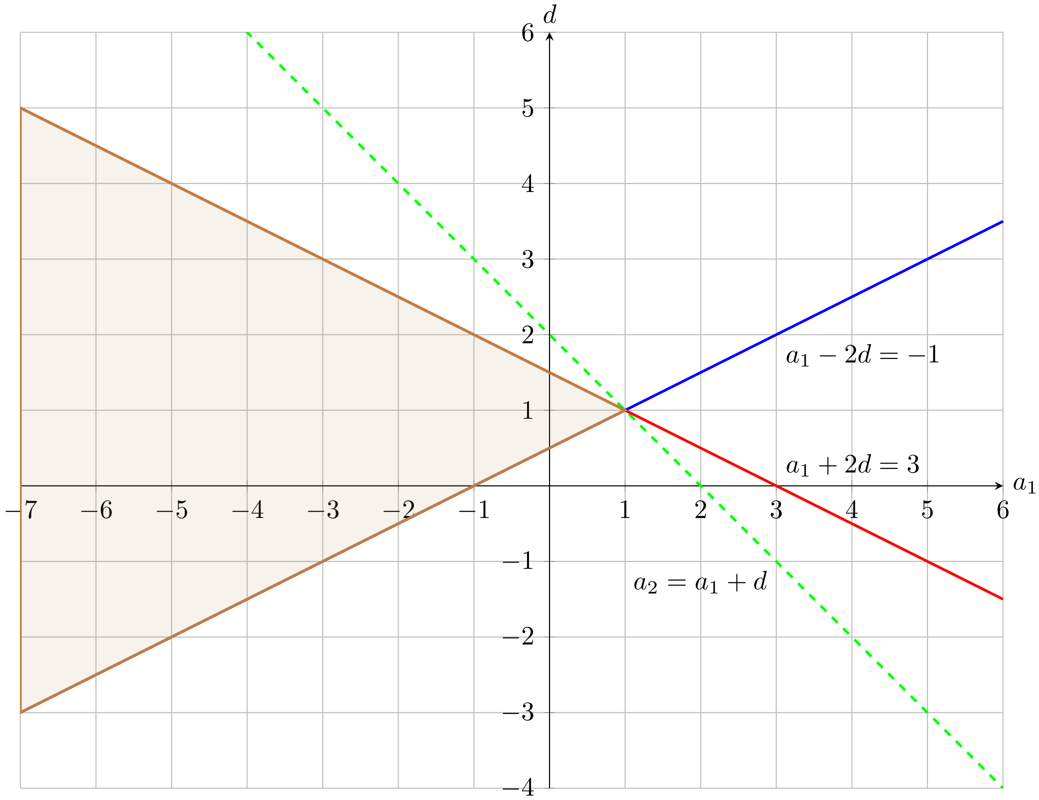
习题1.3. 数列$\{a_n\}$满足
$$a_1 = \frac{1}{2}, \quad a_{n+1} = \frac{a_n}{2a_n + 3}$$
求$\{a_n\}$的通项公式

解.
$$\begin{gathered} a_{n+1} = \frac{a_n}{2a_n + 3} \\ \Rightarrow \frac{1}{a_{n+1}} = 3\frac{1}{a_n} + 2 \\ \Rightarrow \frac{1}{a_{n+1}} + 1 = 3\left(\frac{1}{a_n} + 1\right) \\ \Rightarrow a_{n+1} = \frac{1}{3^n - 1} \end{gathered}$$

□

习题1.4. 等差数列$\{a_n\}$满足
$$S_3 + S_6 < 27, \quad S_3 - a_1 - a_2 \leq -1$$
求a_2的最大值

解. 设$a_n = a_1 + (n-1)d$
$$\begin{cases} S_3 + S_6 < 27 \\ S_3 - a_1 - a_2 \leq -1 \end{cases} \quad \Rightarrow \quad \begin{cases} a_1 + 2d \leq 3 \\ a_1 - 2d \leq -1 \end{cases}$$
根据线性规划知识，我们知道a_2的最大值为2



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习题1.5. 数列$\{a_n\}$满足$a_n = n2^{n-1}$，试求S_n

解.

$$S_n = 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 2^2 + \cdots + n \cdot 2^{n-1} \tag{1.1}$$

$$2S_n = 1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + n \cdot 2^n \tag{1.2}$$

(1.2)式减去(1.1)式，得

$$S_n = n2^n - 1 - (2 + 2^2 + 2^3 + \cdots + 2^{n-1}) = (n-1)2^n + 1$$

□

习题1.6. 数列$\{a_n\}$满足$a_{n+1} = 3a_n + 1, a_1 = 1$，试求通项公式$a_n$

解.

$$\begin{aligned} a_{n+1} + \lambda &= 3a_n + 1 + \lambda \\ &= 3\left(a_n + \frac{1+\lambda}{3}\right) \\ &= 3(a_n + \lambda) \end{aligned}$$

$$\lambda = \frac{1+\lambda}{3} \Rightarrow \lambda = \frac{1}{2}$$

$$a_{n+1} + \frac{1}{2} = 3\left(a_n + \frac{1}{2}\right)$$

$$a_{n+1}+\frac{1}{2}=3^{n-1}\left(a_1+\frac{1}{2}\right)=\frac{3^n}{2}\Rightarrow a_n=\frac{3^n}{2}-\frac{1}{2}$$

□

习题1.7. 斐波那契(Fibonacci)数列满足$a_1=a_2=1$， $a_{n+2}=a_{n+1}+a_n$，试求其通项公式

解.

$$\begin{aligned}a_{n+2}-\lambda a_{n+1}&=(1-\lambda)a_{n+1}+a_n\\&=(1-\lambda)\left(a_{n+1}+\frac{1}{1-\lambda}a_n\right)\\&=(1-\lambda)(a_{n+1}-\lambda a_n)\end{aligned}$$

$$-\lambda=\frac{1}{1-\lambda}\Rightarrow\lambda=\frac{1+\sqrt{5}}{2}$$

$$a_{n+1}-\lambda a_n=(1-\lambda)^{n-1}(a_2-\lambda a_1)=(1-\lambda)^n$$

$$\frac{a_{n+1}}{\lambda^{n+1}}=\frac{a_n}{\lambda^n}+\frac{1}{1-\lambda}\left(\frac{1-\lambda}{\lambda}\right)^{n+1}$$

$$\frac{a_n}{\lambda^n}=\frac{a_1}{\lambda}+\frac{\frac{1-\sqrt{5}}{1+\sqrt{5}}-\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n}{\sqrt{5}}$$

$$a_n=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

□

习题1.8. 数列$\{a_n\}$满足$a_{n+1}=\dfrac{2a_n+1}{a_n+2}, a_1=2$，试求通项公式$a_n$

解.

$$\begin{aligned}a_{n+1}+\lambda&=\frac{(2+\lambda)a_n+(1+2\lambda)}{a_n+2}\\&=(2+\lambda)\frac{a_n+\frac{1+2\lambda}{2+\lambda}}{a_n+2}\\&=(2+\lambda)\frac{a_n+\lambda}{a_n+2}\end{aligned}$$

$$\lambda=\frac{1+2\lambda}{2+\lambda}\Rightarrow\lambda=1$$

$$a_{n+1}+1=3\frac{a_n+1}{a_n+2}$$

$$\frac{1}{a_{n+1}+1}=\frac{1}{3}\frac{1}{a_n+1}+\frac{1}{3}$$

$$\frac{1}{a_{n+1}+1}-\frac{1}{2}=\frac{1}{3}\left(\frac{1}{a_n+1}-\frac{1}{2}\right)$$

$$\frac{1}{a_n+1}-\frac{1}{2}=\frac{1}{3^{n-1}}\left(\frac{1}{a_1+1}-\frac{1}{2}\right)$$

$$a_n=\frac{3^n+1}{3^n-1}$$

□

习题1.9. 数列a_n满足$a_{n+1} = c - \frac{1}{a_n}, a_1 = 1$，若$a_n < a_{n+1} < 3$，求$c$的取值范围

解. 运用不动点的知识

$$a_1 < a_2 = c - \frac{1}{a_1} \Rightarrow c > 2$$

方程$x = c - \frac{1}{x} \iff x^2 - cx + 1 = 0$有两根$\frac{c-\sqrt{c^2-4}}{2} = x_1 < x_2 = \frac{c+\sqrt{c^2-4}}{2}$
则有$x_1 + x_2 = c, x_1x_2 = 1$，故

$$x_2 - a_{n+1} = x_2 - c + \frac{1}{a_n} = \frac{1}{a_n} - x_1 = \frac{1}{a_n} - \frac{1}{x_2} = \frac{x_2 - a_n}{a_n x_2}$$

从函数图象可以猜测$x_2 \le 3$，故c的取值范围应当是$2 < c \le \frac{10}{3}$若$2 < c \le \frac{10}{3}$，使用归纳法有

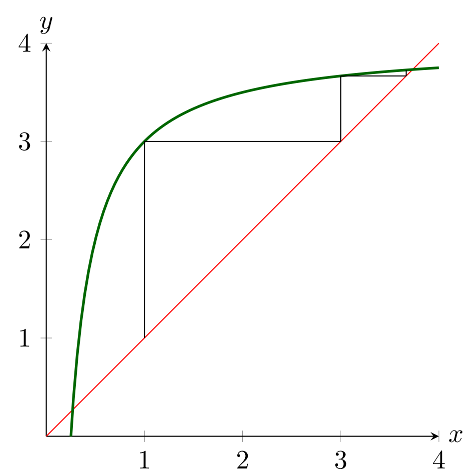
$$a_{n+1} = c - \frac{1}{a_n} < \frac{10}{3} - \frac{1}{3} = 3$$

$$x_2 - a_{n+1} = \frac{x_2 - a_n}{a_n x_2} \le \frac{x_2 - a_n}{x_1 x_2} = x_2 - a_n$$

若$c > \frac{10}{3}$，则$x_2 > 3$，使用归纳法有

$$x_2 - a_{n+1} = \frac{x_2 - a_n}{a_n x_2} \le \frac{x_2 - a_n}{x_2} < \frac{x_2 - a_n}{3}$$

故n当足够大时，a_n和x_2可以任意接近，这与$a_n < 3$矛盾



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2 平面几何

习题2.1. 试证明三角形的三条垂线交于一点

证明. 不妨设三角形的三个顶点分别为$O(0,0), A(a,0), B(b,c) \quad (a,c>0)$
假设点$P=(x,y)$就是要找的垂心

$$\begin{cases}\overrightarrow{OA}\cdot\overrightarrow{BP}=0\\ \overrightarrow{OB}\cdot\overrightarrow{AP}=0\end{cases} \Rightarrow \begin{cases}(x-b)a=0\\ (x-a)b+cy=0\end{cases} \Rightarrow \begin{cases}x=b\\ y=\dfrac{b(a-b)}{c}\end{cases}$$

但与此同时

$$(b-a)x+cy=0 \quad \Rightarrow \quad \overrightarrow{OP}\cdot\overrightarrow{AB}=0$$

□

习题2.2. $\vec{a}=(a_1,a_2,a_3), \vec{b}=(b_1,b_2,b_3)$，试求$\vec{a}\times\vec{b}$

解.

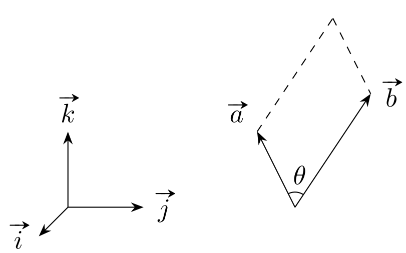
$$\vec{i}\times\vec{j}=\vec{k}, \quad \vec{j}\times\vec{k}=\vec{i}, \quad \vec{k}\times\vec{i}=\vec{j}$$

$$\begin{aligned}\vec{a}\times\vec{b} &= (a_1\vec{i}+a_2\vec{j}+a_3\vec{k})\times(b_1\vec{i}+b_2\vec{j}+b_3\vec{k})\\ &= (a_1b_2-a_2b_1)\vec{i}\times\vec{j}+(a_1b_3-a_3b_1)\vec{i}\times\vec{k}+(a_2b_3-a_3b_2)\vec{j}\times\vec{k}\\ &= (a_2b_3-a_3b_2)\vec{i}-(a_1b_3-a_3b_1)\vec{j}+(a_1b_2-a_2b_1)\vec{k}\end{aligned}$$

特殊的，当$\vec{a}=(a_1,a_2), \vec{b}=(b_1,b_2)$时，可通过此法求出两向量形成平行四边形的面积

$$|\vec{a}|\cdot|\vec{b}|\sin\theta=|\vec{a}\times\vec{b}|=|a_1b_2-a_2b_1|$$

其中θ是向量$\vec{a}, \vec{b}$的夹角



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习题2.3. 设$\overrightarrow{OA}=\vec{a}, \overrightarrow{OB}=\vec{b}$，$N$是$OA$中点，$M$是$AB$的三等分点，$OM, BN$交于点$P$，试求$\overrightarrow{OP}$

解.

$$\overrightarrow{OM}=\frac{1}{3}\vec{a}+\frac{2}{3}\vec{b}, \quad \overrightarrow{BN}=\frac{1}{2}\vec{a}-\vec{b}$$

设$\overrightarrow{OP}=x\vec{a}+y\vec{b}$，由$\overrightarrow{OP}\parallel\overrightarrow{OM}, \quad \overrightarrow{BP}\parallel\overrightarrow{BN}$得

$$\begin{cases}\dfrac{1}{3}y-\dfrac{2}{3}x=0\\ \dfrac{1}{2}(y-1)+x=0\end{cases} \Rightarrow \begin{cases}x=\dfrac{1}{4}\\ y=\dfrac{1}{2}\end{cases}$$

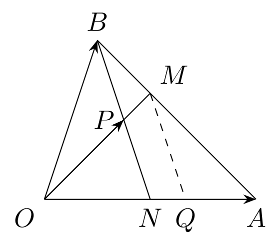
□

解.

$$\frac{AM}{AB} = \frac{AN}{AQ} = \frac{2}{3}$$

$$\Rightarrow \frac{OQ}{ON} = \frac{OM}{OP} = \frac{3}{4}$$

$$\overrightarrow{OP} = \frac{3}{4}\overrightarrow{OM} = \frac{1}{4}\vec{a} + \frac{1}{2}\vec{b}$$



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3 圆锥曲线

习题3.1. $C: y^2 = 4x$是以$(-3,0)$为焦点的抛物线，直线l交C于A, B' B'关于x轴的对称点为B，AB交x轴于M，试求M点坐标

证明. 假设A, B', B, M的坐标分别为$(x_1, y_1), (x_2, y_2), (x_2, -y_2), (x_0, 0)$，直线$l$方程为$y = k(x+3)$，则有

$$k^2x^2 + (6k^2 - 4)x + 9k^2 = 0$$

设x_1, x_2是方程的两根，由韦达(Vièta)定理知

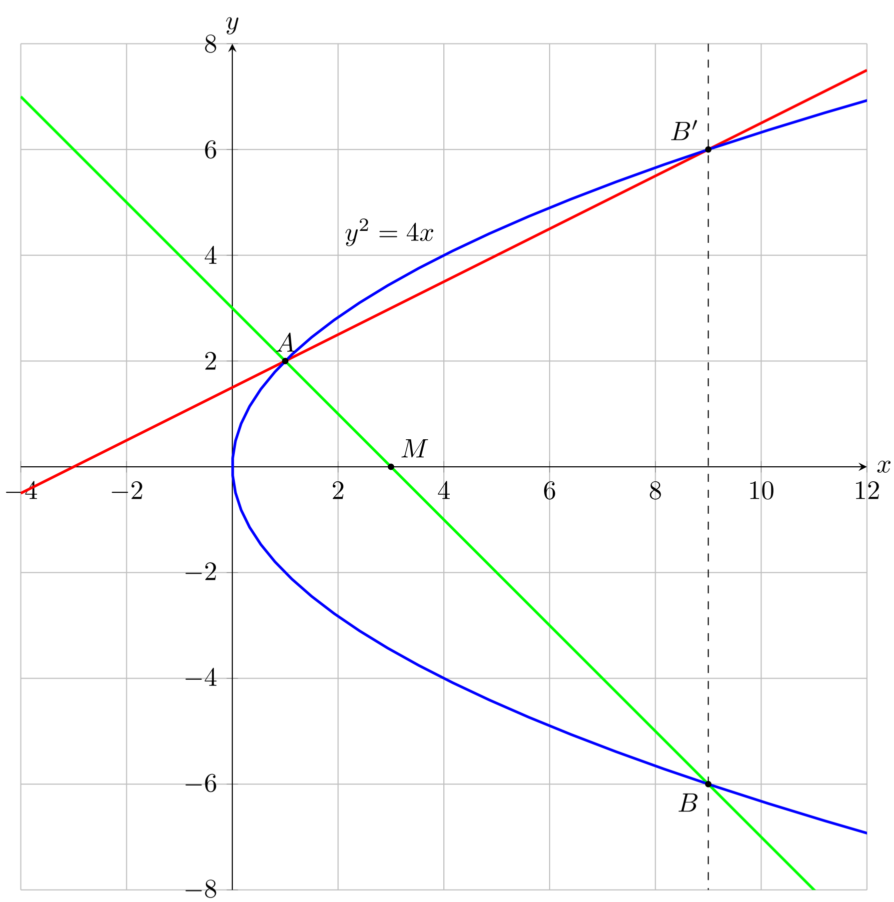
$$\begin{cases} x_1 + x_2 = -\dfrac{6k^2 - 4}{k^2} \\ x_1x_2 = 9 \end{cases}$$

由于AM, BM的斜率相同，故

$$\frac{y_1}{x_1 - x_0} = \frac{-y_2}{x_2 - x_0}$$

$$x_0 = \frac{x_1y_2 + x_2y_1}{y_1 + y_2} = \frac{3k(x_1 + x_2) + 2kx_1x_2}{k(x_1 + x_2 + 6)} = \frac{3(x_1 + x_2) + 2x_1x_2}{x_1 + x_2 + 6}$$

$$x_0 = \frac{3(x_1 + x_2) + 18}{x_1 + x_2 + 6} = 3$$



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习题3.2. 直线$l: y=kx+m$交椭圆$C: \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$于$A,B$，试求$|AB|$

解. 设点A,B的坐标分别为$(x_1,y_1),(x_2,y_2)$

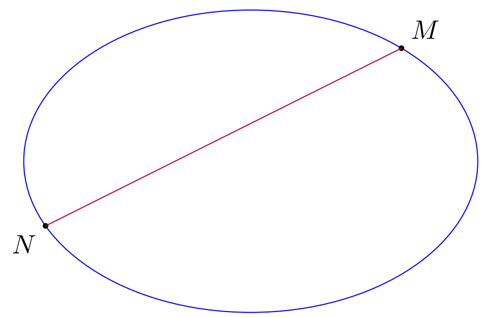
$$(a^2k^2+b^2)x^2+2kma^2x+a^2(m^2-b^2)=0$$

设x_1,x_2是方程的两根，由韦达(Vièta)定理知

$$\begin{cases} x_1+x_2=-\dfrac{2kma^2}{a^2k^2+b^2} \\ x_1x_2=\dfrac{a^2(m^2-b^2)}{a^2k^2+b^2} \end{cases}$$

$$\begin{aligned} |AB| &= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \\ &= \sqrt{(x_1-x_2)^2+k^2(x_1-x_2)^2} \\ &= \sqrt{1+k^2}\cdot\sqrt{(x_1-x_2)^2} \\ &= \sqrt{1+k^2}\cdot\sqrt{(x_1+x_2)^2-4x_1x_2} \end{aligned}$$

$$|AB|=\frac{2ab}{a^2k^2+b^2}\sqrt{(1+k^2)(a^2k^2+b^2-m^2)}$$



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习题3.3. 过$P=(0,2)$的直线l交椭圆$C: \dfrac{x^2}{2}+y^2=1$于M,N，试求$S_{\Delta OMN}$

解. 设A,B坐标分别为$(x_1,y_1),(x_2,y_2)$，l方程为$y=kx+2$，则有

$$(2k^2+1)x^2+8kx+6=0$$

$$\Delta=16k^2-24>0 \Rightarrow k^2>\frac{3}{2}$$

设x_1,x_2是方程的两根，由韦达(Vièta)定理知

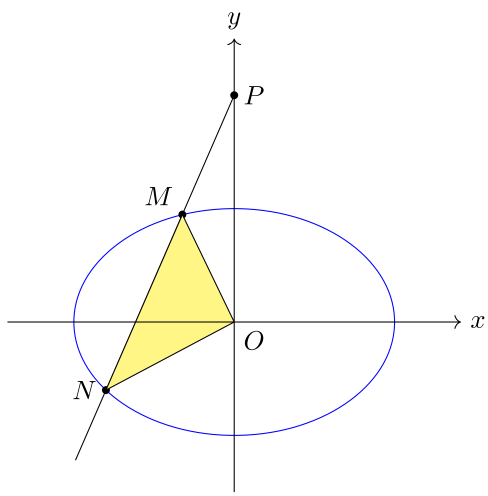
$$\begin{cases} x_1+x_2=-\dfrac{8k}{2k^2+1} \\ x_1x_2=\dfrac{6}{2k^2+1} \end{cases}$$

$$
\begin{aligned}
S_{\Delta OMN} &= \frac{1}{2}|\overrightarrow{OM} \times \overrightarrow{ON}| \\
&= \frac{1}{2}|x_1y_2 - x_2y_1| \\
&= |x_1 - x_2| \\
&= \sqrt{(x_1+x_2)^2 - 4x_1x_2} \\
&= \frac{2\sqrt{4k^2-6}}{2k^2+1}
\end{aligned}
$$

令$u = \sqrt{4k^2-6}$

$$
S_{\Delta OMN} = \frac{4u}{u^2+8} = \frac{4}{u + \dfrac{8}{u}}
$$

当$k = \pm\dfrac{\sqrt{14}}{2}$即$u = 2\sqrt{2}$时$S_{\Delta OMN}$取最小值$\dfrac{\sqrt{2}}{2}$



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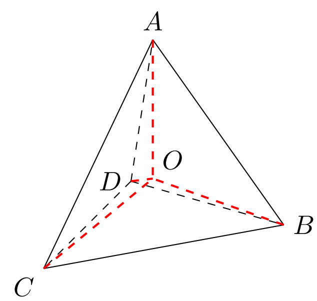
4 立体几何

习题4.1. $ABCD$是正四面体，O为其中心，试求$\overrightarrow{OA},\overrightarrow{OB}$之间的夹角的余弦值

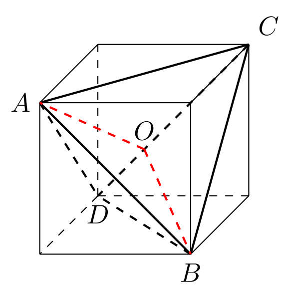
解. 设正方体边长为1

$$|OA| = |OB| = \frac{\sqrt{3}}{2}, \quad |AB| = \sqrt{2}$$

$$\cos\langle\overrightarrow{OA},\overrightarrow{OB}\rangle = \frac{|OA|^2 + |OB|^2 - |AB|^2}{2|OA||OB|} = -\frac{1}{3}$$



可以把正四面体放进正方体中来看



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5 导数与积分

习题5.1. 已知$(e^x)' = e^x$，求$(a^x)'(a > 0)$

解. 由于$a > 0$，令$a = e^k$，有$k = \ln a$

$$(a^x)' = (e^{kx})' = ke^{kx} = a^x \ln a$$

□

习题5.2. 求证$(x^n)' = nx^{n-1}$

证明.

$$\begin{aligned}
(x^n)' &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{C_n^0 x^n + C_n^1 x^{n-1}(\Delta x) + C_n^2 x^{n-2}(\Delta x)^2 + \cdots + C_n^n (\Delta x)^n - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} C_n^1 x^{n-1} + C_n^2 x^{n-2}(\Delta x) + \cdots + C_n^n (\Delta x)^{n-1} \\
&= nx^{n-1}
\end{aligned}$$

注:

$$(x^\alpha)' = (e^{\alpha \ln x})' = e^{\alpha \ln x} \frac{\alpha}{x} = \alpha x^{\alpha - 1}$$

□

6 排列组合

7 三角函数

习题7.1. 求$\sin\theta+\sin 2\theta+\cdots+\sin n\theta$

解.

$$\begin{aligned}
&2\sin\frac{\theta}{2}(\sin\theta+\sin 2\theta+\cdots+\sin n\theta)\\
=&2\sin\theta\sin\frac{\theta}{2}+2\sin 2\theta\sin\frac{\theta}{2}+\cdots+2\sin n\theta\sin\frac{\theta}{2}\\
=&(\cos\frac{\theta}{2}-\cos\frac{3}{2}\theta)+(\cos\frac{3}{2}\theta-\cos\frac{5}{2}\theta)+\cdots+(\cos\frac{2n-1}{2}\theta-\cos\frac{2n+1}{2}\theta)\\
=&\cos\frac{\theta}{2}-\cos\frac{2n+1}{2}\theta
\end{aligned}$$

$$\sin\theta+\sin 2\theta+\cdots+\sin n\theta=\frac{\cos\frac{\theta}{2}-\cos\frac{2n+1}{2}\theta}{2\sin\frac{\theta}{2}}$$

□

习题7.2. 试求$\sin 10^\circ\sin 30^\circ\sin 50^\circ\sin 70^\circ$

解.

$$\begin{aligned}
\sin 10^\circ\sin 30^\circ\sin 50^\circ\sin 70^\circ&=\frac{1}{2}\cos 80^\circ\cos 40^\circ\cos 20^\circ\\
&=\frac{\cos 80^\circ\cos 40^\circ\cos 20^\circ\sin 20^\circ}{2\sin 20^\circ}\\
&=\frac{\cos 80^\circ\cos 40^\circ\sin 40^\circ}{4\sin 20^\circ}\\
&=\frac{\cos 80^\circ\sin 80^\circ}{8\sin 20^\circ}\\
&=\frac{\sin 160^\circ}{16\sin 20^\circ}\\
&=\frac{1}{16}
\end{aligned}$$

□

习题7.3. 化简$A\sin\omega x+B\cos\omega x$

解.

$$\begin{aligned}
A\sin\omega x+B\cos\omega x&=\sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\sin\omega x+\frac{B}{\sqrt{A^2+B^2}}\cos\omega x\right)\\
&=\sqrt{A^2+B^2}\,(\cos\varphi\sin\omega x+\sin\varphi\cos\omega x)\\
&=\sqrt{A^2+B^2}\sin(\omega x+\varphi)
\end{aligned}$$

$\cos\varphi=\dfrac{A}{\sqrt{A^2+B^2}},\quad\sin\varphi=\dfrac{B}{\sqrt{A^2+B^2}},\quad\tan\varphi=\dfrac{B}{A}$ □

习题7.4. 已知$\tan\theta$，试求$\sin^2\theta+\sin\theta\cos\theta$

解.

$$\sin^2\theta + \sin\theta\cos\theta = \frac{\sin^2\theta + \sin\theta\cos\theta}{\sin^2\theta + \cos^2\theta} = \frac{\tan^2\theta + \tan\theta}{\tan^2\theta + 1}$$

□

习题7.5. 求证$\dfrac{\sin\alpha}{1+\cos\alpha} = \dfrac{1-\cos\alpha}{\sin\alpha}$

证明.

$$\frac{\sin\alpha}{1+\cos\alpha} = \frac{2\sin\dfrac{\alpha}{2}\cos\dfrac{\alpha}{2}}{2\cos^2\dfrac{\alpha}{2}} = \tan\frac{\alpha}{2} = \frac{2\sin^2\dfrac{\alpha}{2}}{2\sin\dfrac{\alpha}{2}\cos\dfrac{\alpha}{2}} = \frac{1-\cos\alpha}{\sin\alpha}$$

□

证明.

$$1-\cos^2\alpha = \sin^2\alpha \Rightarrow \frac{\sin\alpha}{1+\cos\alpha} = \frac{1-\cos\alpha}{\sin\alpha}$$

□

8 函数

习题8.1. 求证任意一个函数可以写作一个偶函数和一个奇函数的和

解. 设$h(x) = f(x) + g(x)$，其中$f(x)$是偶函数，$g(x)$是奇函数

$$\begin{cases} f(x) + g(x) = h(x) \\ f(x) - g(x) = h(-x) \end{cases} \quad \Rightarrow \quad \begin{cases} f(x) = \dfrac{h(x) + h(-x)}{2} \\ g(x) = \dfrac{h(x) - h(-x)}{2} \end{cases}$$

□

9 不等式

习题9.1. 试解不等式 $\dfrac{2^2+x+1}{x^2-5x+6} \geq 3$

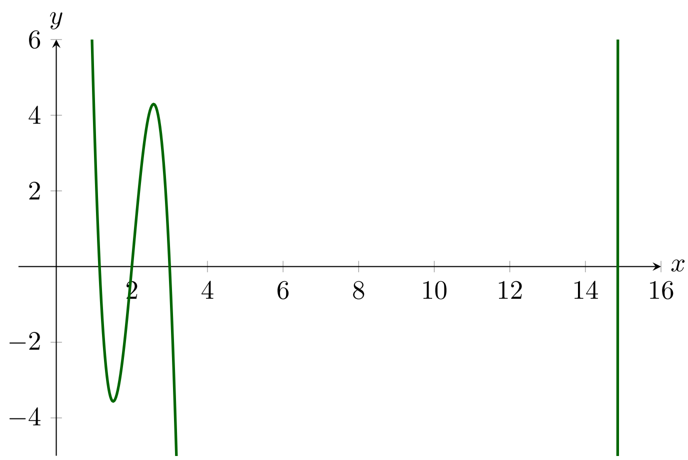
解.

$$\frac{x^2+x+1}{x^2-5x+6} \geq 3 \Rightarrow \frac{x^2-16x+17}{(x-2)(x-3)} \geq 0$$

$$\Rightarrow \begin{cases} (x^2-16x+17)(x-2)(x-3) \geq 0 \\ x \neq 2, \quad x \neq 3 \end{cases}$$

$$\Rightarrow \begin{cases} \left(x-(8-\sqrt{47})\right)(x-2)(x-3)\left(x-(8+\sqrt{47})\right) \geq 0 \\ x \neq 2, \quad x \neq 3 \end{cases}$$

$$\Rightarrow x \in (-\infty, 8-\sqrt{47}] \cup (2,3) \cup [8+\sqrt{47}, +\infty)$$



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