ABSTRACT

Title of dissertation: HOPF ALGEBRA OF MULTIPLE POLYLOGARITHMS AND ASSOCIATED MIXED HODGE STRUCTURES

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This thesis constructs a variation of mixed Hodge structures based on multiple polylogarithms, and attempts to build candidate complexes for computing motivic cohomology.

Firstly, we consider Hopf algebras with generators representing multiple polylogarithms. By quotienting products and functional relations, we get Lie coalgebras whose Chevalley-Eilenberg complexes are conjectured to compute rational and integral motivic cohomologies. We also associate one-forms to multiple polylogarithms, which exhibit combinatorial properties that are easy to work with. This is joint work with Greenberg, Kaufman and Zickert, which constitues [1] and part of [2].

Next, we introduce a variation matrix which describes a variation of mixed Hodge structures encoded by multiple polylogarithms. Its corresponding connection form is composed of the one-forms associated to the multiple polylogarithms. This joint work constitutes the remaining part of [2].

Lastly, to ensure the well-definedness of the Hodge structures, we must compute

the monodromies of multiple polylogarithms, for which we provide an explicit formula, extending the previous work done for multiple logarithms, a subfamily of multiple polylogarithms.

HOPF ALGEBRA OF MULTIPLE POLYLOGARITHMS AND ASSOCIATED MIXED HODGE STRUCTURES

by

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List of Abbreviations and Symbols

$\mathrm{Li}_{n_1,\cdots,n_d}(x_1,\cdots,x_d)$	Standard multiple polylogarithms
$\mathcal{L}_{n_1,\cdots,n_d}(x_1,\cdots,x_d)$	Single-valued multiple polylogarithms
$\widehat{\mathcal{L}}_{n_1,\cdots,n_d}$	Lifted multiple polylogarithms
$S_d(\mathbb{C})$	Domain for multiple polylogarithms of depth d
$\widetilde{S}_d(\mathbb{C})$	Universal cover of $S_d(\mathbb{C})$
$\widehat{S}_d(\mathbb{C})$	Universal abelian cover of $S_d(\mathbb{C})$
$\widehat{\mathbb{C}}$	Universal abelian cover of $\mathbb{P}^1-\{0,1,\infty\}$ or $\widehat{S}_1(\mathbb{C})$
$I_\gamma(a_0;\cdots;a_{n+1})$	Iterated integral of logarithmic differentials $\frac{dz}{z-a_i}$
$H^i_{\mathcal{M}}(F,\mathbb{Z}(n))$	Motivic cohomology of a field F
$\Delta_{1,\cdots,1}$	Iterated coproduct or the symbol map
$x^{\pm}_{i\to j}$	Consecutive product $(x_i\cdots x_{j-1})^{\pm}$
0^k	A tuple of k zeros
$\widetilde{I}(S)$	Hopf algebra of symbolic iterated integrals with arguments in S
$I(S)$	$\widetilde{I}(S)$ modulo degenerates
$\mathbb{I}^{\mathrm{Symb}}$	Hopf algebra generated by iterated integral symbols
$\mathbb{H}^{\mathrm{Symb}}$	Hopf algebra generated by multiple polylogarithms symbols
$\mathbb{L}^{\mathrm{Symb}}$	Lie coalgebra by modulo products in $\mathbb{H}^{\mathrm{Symb}}$
$\overline{\mathbb{H}}^{\mathrm{Symb}}$	$\mathbb{H}^{\mathrm{Symb}}$ ajoined by inverted multiple polylogarithms symbols
$\widehat{\mathbb{H}}^{\mathrm{Symb}}_M$	Sheafification of morphisms from open subsets of M to $\mathbb{H}^{\mathrm{Symb}}$
$\mathbb{L}_M$	$\widehat{\mathbb{H}}^{\mathrm{Symb}}_M$ modulo by products and functional relations on M
$\overline{\mathrm{INV}}$	Goncharov inversion of multiple polylogarithms for $\overline{\mathbb{H}}^{\mathrm{Symb}}$
INV	$\overline{\mathrm{INV}}$ modulo πi
Φ	Map turning symbols in $\mathbb{I}^{\mathrm{Symb}}$ into symbols in $\overline{\mathbb{H}}^{\mathrm{Symb}}$
w	One-form map that associates one-forms to multiple polylogarithms
V^H	Variation matrix associated to a free contraction Hopf algebra H
$V^H_{p,q}$	The (p,q)-th weight block of V^H
V^H_n	Sum of $V^H_{p,q}$ such that $p+q=n$
$\Re^H$	Realization of symbols in $H=\mathbb{I}^{\mathrm{Symb}},\overline{\mathbb{H}}^{\mathrm{Symb}},\mathbb{H}^{\mathrm{Symb}}$ as functions
$\tau_{n_1,\cdots,n_d}(a)$	Diagonal matrix with entries powers of a according to the gradings
$I^w(a_{i_0};\cdots;a_{i_{n+1}})$	Splits $I(a_{i_0};\cdots;a_{i_{n+1}})$ as products according to the letters in w
$\widehat{V}$	Lifted variation matrix made out of lifted multiple polylogarithms
$\widehat{\nabla}$	Connection on the lifted variation of mixed Hodge structures
$\widehat{\omega}$	Connection form for the lifted variation of mixed Hodge structures
$\mathcal{M}_{\nu_i}$	Monodromy operator for iterated integrals around $x_i=0$
$\mathcal{M}_{\nu_{j,k}}$	Monodromy operator for iterated integrals around $x_j\cdots x_k=1$

Chapter 1: Introduction

1.1 Background and motivation

Polylogarithms, denoted as $\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$, are a collection of multi-valued holomorphic functions on $\mathbb{C} - \{0, 1\}$, extended through analytic continuation. They are named polylogarithms because $\mathrm{Li}_1(z) = -\log(1-z)$, thereby generalizing the natural logarithm. One could also define their single-valued variants $\mathcal{L}_n(z)$ (see Definition 4.3.1). Polylogarithms satisfy various functional equations, including the classical five-term relation

$$\mathcal{L}_2(x) - \mathcal{L}_2(y) + \mathcal{L}_2\left(\frac{y}{x}\right) - \mathcal{L}_2\left(\frac{1-x^{-1}}{1-y^{-1}}\right) + \mathcal{L}_2\left(\frac{1-x}{1-y}\right) = 0 \tag{1.1.1}$$

Polylogarithms play a significant role in several areas of mathematics and physics. They appear in formulas for scattering amplitudes in quantum field theory [3]. Moveover, $\mathcal{L}_2$ shows up in the formula for volumes of hyperbolic 3-simplices [4], and Cheeger-Chern-Simons class for $\mathrm{SL}(2, \mathbb{C})$ [5].

Polylogarithms naturally give rise to a variation of mixed Hodge structures over $\mathbb{P}^1 - \{0, 1, \infty\}$ [6]. Beilinson and Deligne constructed a matrix (see 4.0.1) composed of polylogarithms, which we shall refer to as the variation matrix. The variation matrix is the fundamental matrix to a linear differential equation defined on $\mathbb{P}^1 - \{0, 1, \infty\}$.

They calculated its monodromy matrices, and showed that filtrations on its columns define a variation of mixed Hodge structures.

Later, Goncharov made connections between polylogarithms and the conjectural category of mixed Tate motives over a number field F [7]. In particular, he constructed groups $\mathcal{B}_n(F) = \mathbb{Z}[\mathbb{P}^1_F]/R_n(F)$, where each generator $[a]_n$ can be viewed as representing $\mathcal{L}_n(a)$, and $R_n(F)$ is a subgroup of $\mathbb{Z}[\mathbb{P}^1_F]$ defined inductively, and may be thought of as generated by functional relations for $\mathcal{L}_n$. For example, $R_2(F)$ is widely believed to be generated by all five-term relations

$$[x]_2 - [y]_2 + \left[\frac{y}{x}\right]_2 - \left[\frac{1-x^{-1}}{1-y^{-1}}\right]_2 + \left[\frac{1-x}{1-y}\right]_2 \tag{1.1.2}$$

which corresponds to (1.1.1). These $\mathcal{B}$ groups fit into a chain complex $\Gamma(F,n)$, which reads: (See (3.2.1) for the definitions of the differentials)

$$\mathcal{B}_n(F) \xrightarrow{\delta_n} \mathcal{B}_{n-1}(F) \otimes F^\times \xrightarrow{\delta_{n-1}} \mathcal{B}_{n-2}(F) \otimes \bigwedge^2 F^\times \to \cdots \xrightarrow{\delta_2} \bigwedge^n F^\times \tag{1.1.3}$$

Goncharov conjectured that the i-th cohomology group of $\Gamma(F,n)$ is rationally isomorphic to the i-th motivic cohomology group $H^i_{\mathcal{M}}(F, \mathbb{Q}(n))$. Notice that this is only true rationally, and it fails to be true integrally even for $n = 2$.

There is also a multivariate analog of polylogarithm, called multiple polylogarithms, introduced by Goncharov in [8]. The multiple polylogarithm $\mathrm{Li}_{n_1,\cdots,n_d}(x_1,\cdots,x_d)$ is defined as the power series

$$\sum_{0<k_1<\cdots<k_d} \frac{x_1^{k_1}\cdots x_d^{k_d}}{k_1^{n_1}\cdots k_d^{n_d}}$$

for $|x_i| < 1$. We refer to d and $|\mathbf{n}| = n_1 + \cdots + n_d$ as its *depth* and *weight*. Note that multiple polylogarithms in depth 1 are the classical polylogarithms. Extended by analytic continuation, they are multi-valued holomorphic functions on a subvariety $S_d(\mathbb{C})$ of $\mathbb{C}^d$ by removing the locus of $\{x_i = 0\}_i$ and $\{x_j x_{j+1} \cdots x_k = 1\}_{j \le k}$ (see Definition 2.3.1). It is tempting to ask the following questions:

Question 1.1.1.

i. Do the multiple polylogarithms also define variations of mixed Hodge structures encoded by multiple polylogarithms? And how do we compute the monodromies of multiple polylogarithms?

ii. Is there an analog of Goncharov's Bloch complex (1.1.3) with generators representing multiple polylogarithms, which computes the rational motivic cohomology?

The first question is partially addressed by Zhao [9] in the special case of multiple logarithms, which are multiple polylogarithms with all indices equal to one. The complex in the second question is supposed to be quasi-isomorphic to the Chevalley-Eilenberg complex of the motivic Lie coalgebra associated to the motivic Galois group [7]. Goncharov and Rudenko constructed such complexes up to weight 4 using motivic correlators (see [10]). In contrast, our approach avoids motivic correlators, relying simply on symbolic generators of multiple polylogarithms, making the complexes applicable to arbitrary weights (see [1]). Later, Rudenko and Matveiakin extended Goncharov and Rukendo's work to handle all weights as well (see [11]).

In [12], Zickert considered lifted polylogarithms $\widehat{\mathcal{L}}_n$ as functions from $\widehat{\mathbb{C}}$ to $\mathbb{C}/\frac{(2\pi i)^n}{(n-1)!}\mathbb{Z}$, where $\widehat{\mathbb{C}}$ is the universal abelian cover of $\mathbb{P}^1 - \{0, 1, \infty\}$, modeled by

$$\widehat{\mathbb{C}} = \left\{(u, v) \in \mathbb{C}^2 \middle| e^u + e^v = 1\right\} \tag{1.1.4}$$

here u, v should be thought of as $\log(x)$ and $\log(1-x)$, respectively. The term "lifted" refers to constructions over $\widehat{\mathbb{C}}$. Goncharov previously argued that $\mathcal{L}_n$, when viewed as a map $\mathcal{B}_n(\mathbb{C}) \to \mathbb{R}$ by taking $[a]_n$ to $\mathcal{L}_n(a)$, should make the following diagram commutes

$$\begin{tikzcd}
 & H^1_{\mathcal{M}}(\mathbb{C}, \mathbb{Z}(n)) \arrow[r, "b_n"] \arrow[d, "\cong_{\mathbb{Q}}"] & \mathbb{C}/(2\pi i)^n\mathbb{Z} \arrow[d, "\mathrm{ReIm}"] \\
\ker(\mathcal{B}_n(\mathbb{C}) \arrow[r, "\delta_n"] & \mathcal{B}_{n-1} \otimes \mathbb{C}^\times) \arrow[r, "\mathcal{L}_n"] & \mathbb{R}
\end{tikzcd}$$

here b_n is the cycle map (3.2.5), the left is the conjectural rational isomorphism, and ReIm is Re when n is odd and Im when n is even. Zickert then constructed the lifted Bloch complex $\widehat{\Gamma}(\mathbb{C}, n)$ by defining $\widehat{\mathcal{B}}_n(\widehat{\mathbb{C}})$ groups, whose generators are thought to be $\widehat{\mathcal{L}}_n$, and conjectured its i-th cohomology group is isomorphic to the i-th integral motivic cohomology group $H^i_{\mathcal{M}}(\mathbb{C}, \mathbb{Z}(n))$, rather than just $H^i_{\mathcal{M}}(\mathbb{C}, \mathbb{Q}(n))$. He believe that $\widehat{\mathcal{L}}_n$ would correspond to the cycle map b_n in (1.1).

Moreover, Zickert proved (unpublished) that $\widehat{\mathcal{L}}_n$ introduces a lifted variation of mixed Hodge structures over $\widehat{\mathbb{C}}$. And he also defined the lifted Bloch complex $\widehat{\Gamma}(F, n)$ for a field F satisfying certain necessary conditions (see 3.4.2), and he conjectured its i-th cohomology group is isomorphic to the i-th integral motivic cohomology group $H^i_{\mathcal{M}}(F, \mathbb{Z}(n))$. This leads to several natural questions:

Question 1.1.2.

i. Can we define lifted multiple polylogarithms $\widehat{\mathcal{L}}_{n_1,\cdots,n_d}$ on $\widehat{S}_d(\mathbb{C})$, the universal abelian cover of the domain $S_d(\mathbb{C})$ of $\mathrm{Li}_{n_1,\cdots,n_d}$?

ii. Do these lifted multiple polylogarithms yield a lifted variation of mixed Hodge structures?

iii. Is it possible to construct a lifted motivic complex with generators representing lifted multiple polylogarithms that computes integral (rather than rational) motivic cohomology?

In the thesis, we will try to answer both Question 1.1.1 and Question 1.1.2. The treatment of Question 1.1.1 ii. constitutes the joint work with Greenberg, Kaufman and Zickert [1], and is discussed in Chapter 3. Meanwhile, Question 1.1.2 iii., Question 1.1.2 i. and ii. are addressed in the joint work with Greenberg, Kaufman and Zickert [2], and are covered in Chapter 4. Monodromy calculations for multiple polylogarithms related to Question 1.1.2 iii. is the independent work of the author, which is detailed in Chapter 5.

1.2 Structure of the paper and main results

In Chapter 2, we review fundamental concepts and well-known theorems that set the stage for the discussions in subsequent Chapters. We begin by examining key concepts such as iterated integrals, Hopf algebras, Lie coalgebras and variations of mixed Hodge structures. In particular, we discuss the regularization of singular iterated integrals, where shuffle algebras and Lyndon words play a crucial role. After establishing these foundations, we formally define multiple polylogarithms, explore their key properties, and introduce a coproduct structure on the Hopf algebra of iterated integrals. Additionally, we cover basic definitions and facts regarding connections on vector bundles, and variations of mixed Hodge structures, which prepare us for the discussions in Chapter 4. Lastly, we briefly touch on the connections between multiple polylogarithms and motivic cohomology, further motivating our research.

Chapter 3 is joint work with Zachary Greenberg, Dani Kaufman and Christian K. Zickert [1]. Firstly, we define a Hopf algebra $\mathbb{H}^{\mathrm{Symb}}$ generated by generators $[x_1, \cdots, x_d]_{n_1,\cdots,n_d}$ that represent $\mathrm{Li}_{n_1,\cdots,n_d}(x_1, \cdots, x_d)$. Unlike previous constructions by Goncharov and others, we do not impose shuffle or inversion relations on the generators. This explains the use of the superscript "Symb", indicating that the generators are symbolic, with no relations imposed. We also equip $\mathbb{H}^{\mathrm{Symb}}$ with a natural coproduct derived from Goncharov's coproduct on iterated integrals. Next, we associate differential one-forms to elements in $\mathbb{H}^{\mathrm{Symb}}$. These one-forms live in $\Omega^1_{\mathbb{Q}[\{u_i,v_{j,k}\}]/\mathbb{Q}}$, where u_i, $v_{j,k}$ represent $\log(x_i)$, $\log(1-x_j \cdots x_k)$, which are coordinates on $\widehat{S}_d(\mathbb{C})$ (see (2.3.2)). For instance, the one-form associated to $[x_1]_2$ is $\frac{1}{2}(u_1 dv_{1,1} - v_{1,1} du_1)$. The one-forms offer a new approach to exploring functional relations in addition to another widely used tool, the symbols of multiple polylogarithms [13]. Additionally, we consider several useful Hopf algebras modeled on $\mathbb{H}^{\mathrm{Symb}}$, such as a sheaf of Hopf algebras $\widehat{\mathbb{H}}^{\mathrm{Symb}}_M$ over a complex manifold M, by sheafifying morphisms from open subsets of M to $\widehat{S}_d(\mathbb{C})$. These various Hopf algebras exhibit similar structural properties. To capture the essence of these structures, we introduce the notion of a contraction system and use it to justify all constructions simultaneously. Finally, by taking quotient of $\mathbb{H}^{\mathrm{Symb}}$ by products (see Theorem 2.2.19), we obtain a Lie coalgebra $\mathbb{L}^{\mathrm{Symb}}$, whose Chevalley-Eilenberg complex $\bigwedge^\bullet \mathbb{L}^{\mathrm{Symb}}$ maps to the de Rham complex by sending an elements to its one-form, resulting in a commutative diagram.

Theorem. Suppose $\Omega^\bullet_{\mathbb{Q}[\{u_i,v_{j,k}\}]/\mathbb{Q}}$ is the de Rham complex, and w is the map that takes an element to its one-form, then the following diagram commutes:

$$\begin{tikzcd}
\mathbb{L}^{\mathrm{Symb}} \arrow[r] \arrow[d, "w"] & \bigwedge^2 \mathbb{L}^{\mathrm{Symb}} \arrow[r] \arrow[d, "w\wedge w"] & \bigwedge^3 \mathbb{L}^{\mathrm{Symb}} \arrow[r] \arrow[d, "w\wedge w\wedge w"] & \cdots \\
\Omega^1 \arrow[r, "d"] & \Omega^2 \arrow[r, "d"] & \Omega^3 \arrow[r, "d"] & \cdots
\end{tikzcd}$$

A sheaf of graded Lie coalgebra, denoted $\widehat{\mathbb{L}}_M$, should exist by taking the quotient of $\widehat{\mathbb{H}}_M^{\mathrm{Symb}}$ by the products and functional relations on its stalks over M. Moreover, we conjecture the existence of a chain map $\bigwedge^\bullet \widehat{\mathbb{L}}_M \to \Omega^\bullet$, induced by the theorem above, which could establish a connection between motivic cohomology and singular cohomology.

Conjecture 1.2.1. The following diagram commutes

$$\begin{tikzcd}
H^i(M, (\bigwedge^* \widehat{\mathbb{L}}_M)_n) \arrow[r] & H^i(M, \Omega^*) \arrow[d, "\cong"] \\
H^i_{\mathcal{M}}(M, \mathbb{Z}(n)) \arrow[u, dashed] \arrow[r] & H^i(M, \mathbb{C})
\end{tikzcd}$$

Here subscript n means degree n part (see Definition 2.2.15), for example, if $n = 4$, then $(\bigwedge^* \widehat{\mathbb{L}}_M)_4$ reads

$$(\widehat{\mathbb{L}}_M)_4 \to (\widehat{\mathbb{L}}_M)_3 \otimes (\widehat{\mathbb{L}}_M)_1 \oplus (\widehat{\mathbb{L}}_M)_2 \wedge (\widehat{\mathbb{L}}_M)_2 \to (\widehat{\mathbb{L}}_M)_2 \otimes \bigwedge^2 (\widehat{\mathbb{L}}_M)_1 \to \bigwedge^4 (\widehat{\mathbb{L}}_M)_1$$

where $(\widehat{\mathbb{L}}_M)_k$ is the degree k part of $\widehat{\mathbb{L}}_M$. And $H^i(M, (\bigwedge^* \widehat{\mathbb{L}}_M)_n)$ is the hypercohomology of M with coefficients in the complex $(\bigwedge^* \widehat{\mathbb{L}}_M)_n$. The top arrow is induced by the chain map $\bigwedge^\bullet \widehat{\mathbb{L}}_M \to \Omega^\bullet$. The bottom arrow represents the realization functor from integral motivic cohomology to singular cohomology, while the right arrow is the isomorphism between de Rham cohomology and singular cohomology [14]. The nature of the left arrow is currently unclear and still needs to be determined.

Chapter 4 is partially joint work with Zachary Greenberg, Dani Kaufman and Christian K. Zickert [2]. In this chapter, we aim to address Question 1.1.2 i, ii. Firstly, we introduce a generalized concept of variation matrices, which are naturally decomposed into submatrices of different weights. For example, $V^{\mathbb{H}}$ is the variation matrix entirely composed of generators in $\mathbb{H}$. We also define a realization map $\Re$, which "realizes" these generators as actual multiple polylogarithms, for instance, $\Re([x_1, \cdots, x_d]_{n_1,\cdots,n_d}) = \mathrm{Li}_{n_1,\cdots,n_d}(x_1, \cdots, x_d)$. When $\Re$ is applied to $V^{\mathbb{H}}$, we obtain variation matrices previously considered by Deligne and Zhao, which we denote as V. Next, we show that the columns of V generate the sections of a trivial vector bundle over $S_d(\mathbb{C})$ with the flat connection $\nabla = d - \omega$, where ω is simply the differential of V. Additionally, we explain how filtrations on columns of V define a variation of mixed Hodge structures over $S_d(\mathbb{C})$. Similarly, we define lifted multiple polylogarithms $\widehat{\mathcal{L}}_{n_1,\cdots,n_d}$, and construct a lifted variation matrix $\widehat{V}$ consisting of lifted multiple polylogarithms. Columns of $\widehat{V}$ generate the sections of a trivial vector bundle over $\widehat{S}_d(\mathbb{C})$ with a lifted flat connection $\widehat{\nabla} = d - \widehat{\omega}$, where $\widehat{\omega}$ is composed of one-forms associated to multiple polylogarithms. And filtrations on columns of $\widehat{V}$ define a lifted variation of mixed Hodge structures over $\widehat{S}_d(\mathbb{C})$. Finally, we briefly discuss how to use variation matrices to re-interpret a result by Zickert [12], and explore their connection to the recursion formulas in Greenberg's Thesis [15].

Chapter 5 is the independent work of the author. This chapter contains the proof of the rationality of monodromy of multiple polylogarithms, and explicit formulas and algorithms for calculations of monodromy matrices. Zhao cited a general result in [16] for the existence of rational monodromies, but only gave a closed formulas for monodromies of multiple logarithms $\mathrm{Li}_{1,\cdots,1}$. We propose a new approach to this problem. By carefully deforming the integration path of multiple polylogarithms, we break the problem into canonical subcases, and derive an explicit formula and an algorithm for computing monodromy matrices for an arbitrary multiple polylogarithm. Our construction guarantees that the resulting monodromies are always rational. Furthermore, the algorithm has been implemented in Mathematica by the author.

Chapter 2: Preliminaries

2.1 Iterated integrals and hyperlogarithms

Iterated integrals introduced by Chen [17] possess many appealing combinatorial properties, which will contribute significantly to the construction of $\mathbb{H}^{\mathrm{Symb}}$ later in Chapter 3. We will review these basic properties. In particular, Proposition 2.1.13 states how the evaluation of iterated integrals depends on paths. This proposition is crucial in understanding and computing monodromies of multiple polylogarithms disscussed in Chapter 5.

2.1.1 Iterated integrals

Definition 2.1.1. Suppose f_i are complex-valued continuous functions on $[a, b]$. The *iterated integral* of $f_1, \cdots, f_n$ is inductively defined by

$$\int_a^b f_1(t)dt \cdots f_n(t)dt = \int_a^b \left(\int_a^t f_1(s)ds \cdots f_{n-1}(s)ds \right) f_n(t)dt \tag{2.1.1}$$

Remark 2.1.2. Note that this is not to be confused with repeated iterated integrals in classical calculus.

The concept of iterated integrals can be straightforwardly extended to the context of complex manifolds.

Definition 2.1.3. Suppose X is a complex manifold, and ω_i are holomorphic one-forms. Let γ be a piecewise smooth path on X. The *iterated integral* of $\omega_1, \cdots, \omega_n$ along γ is defined as

$$\int_\gamma \omega_1 \cdots \omega_n = \int_{0 \leq t_1 \leq \cdots \leq t_n \leq 1} \gamma^* \omega_1(t_1) \wedge \cdots \wedge \gamma^* \omega_n(t_n) \tag{2.1.2}$$

In particular, we adopt the convention that if $n = 0$, the value of the iterated integral is defined to be 1.

Example 2.1.4. One can easily show by induction that

$$\int_a^b \overbrace{\frac{dt}{t} \cdots \frac{dt}{t}}^{n} = \int_a^b \overbrace{d\log t \cdots d\log t}^{n} = \frac{1}{n!}(\log b - \log a)^n \tag{2.1.3}$$

Example 2.1.5. Polylogarithms can be expressed as iterated integrals

$$\int_0^x \frac{dt}{1-t} \overbrace{\frac{dt}{t} \cdots \frac{dt}{t}}^{n-1} = \mathrm{Li}_n(x) \tag{2.1.4}$$

Proposition 2.1.6. [18] Iterated integrals have the following properties.

i. $\int_\gamma \omega_1 \cdots \omega_n$ is independent of the parametrization of γ.

ii. Suppose γ^{-1} is the opposite path of γ, then

$$\int_\gamma \omega_1 \cdots \omega_n = (-1)^n \int_{\gamma^{-1}} \omega_n \cdots \omega_1 \tag{2.1.5}$$

iii. Suppose α, γ are paths such that $\alpha(1) = \beta(0)$, then

$$\int_{\alpha\beta} \omega_1 \cdots \omega_n = \sum_{i=0}^{n} \int_\alpha \omega_1 \cdots \omega_i \int_\beta \omega_{i+1} \cdots \omega_n \tag{2.1.6}$$

iv. Product of iterated integrals satisfies the shuffle product relation

$$\int_\gamma \omega_1 \cdots \omega_n \int_\gamma \omega_{n+1} \cdots \omega_{n+m} = \sum_\sigma \int_\gamma \omega_{\sigma(1)} \cdots \omega_{\sigma(n+m)}. \tag{2.1.7}$$

where σ runs over all permutations of $\{1, 2, \cdots, n+m\}$ with $\sigma^{-1}(1) < \cdots < \sigma^{-1}(n)$ and $\sigma^{-1}(n+1) < \cdots < \sigma^{-1}(n+m)$

v. $\displaystyle\int_\gamma \omega_1 \cdots \omega_n$ only depends on the homotopy class of γ in X if ω_i are closed one forms and $\omega_1 \wedge \omega_2 = \omega_2 \wedge \omega_3 = \cdots = \omega_{n-1} \wedge \omega_n = 0$ ([18], Proposition 3.1).

If we only consider logarithmic differential forms like $d\log(z-a) = \dfrac{dz}{z-a}$, the iterated integral is called a *hyperlogarithm*.

Definition 2.1.7. Suppose D is a divisor with support $|D| = \bigcup_{i=0}^{n+1}\{a_i\} \subset \mathbb{C}$, where $a_0 \neq a_1$, $a_n \neq a_{n+1}$. Let γ be a path from a_0 to a_{n+1} that is disjoint from $|D|$ except endpoints, the *hyperlogarithm* $I_\gamma(a_0; a_1, \cdots, a_n; a_{n+1})$ is then defined to be

$$\int_\gamma d\log(z-a_1)\cdots d\log(z-a_n) \tag{2.1.8}$$

Remark 2.1.8. Due to Proposition 2.1.6 (v.), it is easy to see that (2.1.8) is invariant up to homotopy of γ in $\mathbb{C} - |D|$. We also see by convention that $I_\gamma(a_0; a_{n+1}) = 1$.

When the endpoints of γ is in $|D|$, we need to justify that (2.1.8) is still finite, we first prove the following Lemma.

Lemma 2.1.9. [19] Suppose γ is the straight path from 0 to z and $|z| < \min\limits_{1\le i\le d}|a_i|$. Then the iterated integral $I_\gamma(0; a_1, \overbrace{0, \cdots, 0}^{n_1-1}, \cdots, a_d, \overbrace{0, \cdots, 0}^{n_d-1}; z)$ has the following power series expansion:

$$\sum_{1\le m_1<\cdots<m_d} \frac{(-1)^d}{m_1^{n_1}\cdots m_d^{n_d}} \frac{z^{m_d}}{a_1^{m_1}a_2^{m_2-m_1}\cdots a_d^{m_d-m_{d-1}}} \tag{2.1.9}$$

Proof. First note that $I_\gamma(0; a_1, \overbrace{0, \cdots, 0}^{n_1-1}, \cdots, a_d, \overbrace{0, \cdots, 0}^{n_d-1}; z)$ is equal to

$$\int_0^1 \frac{zdt}{tz-a_1} \overbrace{\frac{dt}{t}\cdots\frac{dt}{t}}^{n_1-1} \cdots \frac{zdt}{tz-a_d} \overbrace{\frac{dt}{t}\cdots\frac{dt}{t}}^{n_d-1}$$

and then we can repeatedly use the fact that

$$\frac{z}{tz-a} = -\sum_{m\geq 1} \frac{z^m}{a^m} t^{m-1}$$

$\square$

Corollary 2.1.10. If we replace 0 with $a_0 \neq a_1$, and $|z-a_0| < \min_{1\leq i\leq d} |z-a_i|$, (2.1.9) becomes $I_\gamma(a_0; a_1, \overbrace{a_0, \cdots, a_0}^{n_1-1}, \cdots, a_d, \overbrace{a_0, \cdots, a_0}^{n_d-1}; z)$ which is equal to

$$\sum_{1\leq m_1<\cdots<m_d} \frac{(-1)^d}{m_1^{n_1}\cdots m_d^{n_d}} \frac{(z-a_0)^{m_d}}{(a_1-a_0)^{m_1}(a_2-a_0)^{m_2-m_1}\cdots(a_d-a_0)^{m_d-m_{d-1}}} \tag{2.1.10}$$

Now we can find a piecewise straight path γ' homotopic to γ that for each segment, the condition for Lemma 2.1.9 is satisfied. By applying Proposition 2.1.6 (ii., iii.), we proved that (2.1.8) is indeed finite.

The value of $I_\gamma(a_0; a_1, \cdots, a_n; a_{n+1})$ depends on the homotopy class of γ and the positions of $a_0, a_1, \cdots, a_{n+1}$, so we think of $I(a_0; a_1, \cdots, a_n; a_{n+1})$ without γ as a multi-valued function on

$$\mathcal{D}_n = \left\{(a_0, \cdots, a_{n+1}) \in \mathbb{C}^{n+2} \middle| a_0 \neq a_1, a_n \neq a_{n+1}\right\} \tag{2.1.11}$$

We would like to show this is actually a holomorphic function. This is a little subtle, so we need to prove the following technical lemma, which is typically ignored by other authors.

Lemma 2.1.11. Suppose $(a_0^0, \cdots, a_{n+1}^0) \in \mathcal{D}_n$, and γ_0 is a piecewise smooth path in $\mathbb{C}$ with $\gamma_0(0) = a_0^0$, $\gamma_0(1) = a_{n+1}^0$ and disjoint with the rest of distinct a_i^0's, For any $(a_0, \cdots, a_{n+1})$ in some small neighborhood

$$U_\epsilon = \{|a_0 - a_0^0| < \epsilon\} \times \cdots \times \{|a_{n+1} - a_{n+1}^0| < \epsilon\}$$

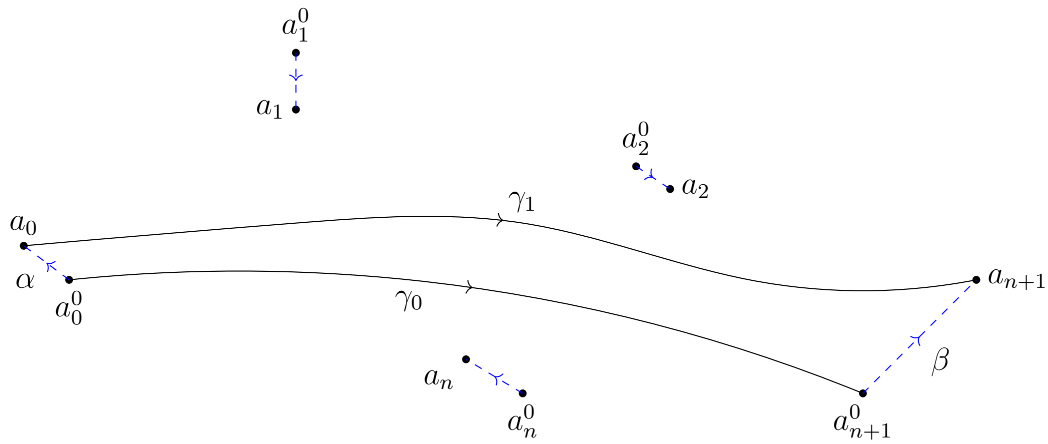


Figure 2.1.1: Analytic continuation of $I(a_0; a_1, \cdots, a_n; a_{n+1})$

of $(a_0^0, \cdots, a_{n+1}^0)$, $I_{\alpha^{-1}\gamma_0\beta}(a_0; a_1, \cdots, a_n; a_{n+1})$ can be written as a power series, where α and β are the straight paths from a_0^0 to a_0 and a_{n+1}^0 to a_{n+1} respectively (see Figure 2.1.1).

Proof. We deploy Proposition 2.1.6 (iii.)

$$I_{\alpha^{-1}\gamma_0\beta}(a_0; a_1, \cdots, a_n; a_{n+1}) = \sum_{0 \le i \le j \le n+1} I_{\alpha^{-1}}(a_0; a_1, \cdots, a_i; a_0^0)$$
$$I_{\gamma_0}(a_0^0; a_{i+1}, \cdots, a_j; a_{n+1}^0) I_\beta(a_{n+1}^0; a_{j+1}, \cdots, a_n; a_{n+1}) \quad (2.1.12)$$

We know that $I_{\gamma_0}(a_0^0; a_{i+1}, \cdots, a_j; a_{n+1}^0)$ is holomorphic in $a_{i+1}, \cdots, a_j$, and according to Proposition 2.1.6 (ii.) and Lemma 2.1.9

$$I_{\alpha^{-1}}(a_0; a_1, \cdots, a_i; a_0^0) = (-1)^i I_\alpha(a_0^0; a_i, \cdots, a_1; a_0), \qquad I_\beta(a_{n+1}^0; a_{j+1}, \cdots, a_n; a_{n+1})$$

can also be written as power series in $a_0, a_1, \cdots, a_i, a_{j+1}, \cdots, a_n, a_{n+1}$. $\square$

Remark 2.1.12. If γ_t is a small continuous deformation of γ_0 such that $|\gamma_t(s) - a_i^0| > \epsilon, \forall a_i^0 \notin \{a_0^0, a_{n+1}^0\}$ and $\gamma_1(0) = a_0$, $\gamma_1(1) = a_{n+1}$, then γ_1 is homotopic to $\alpha^{-1}\gamma_0\beta$ and $I_{\gamma_1}(a_0; a_1, \cdots, a_n; a_{n+1}) = I_{\alpha^{-1}\gamma_0\beta}(a_0; a_1, \cdots, a_n; a_{n+1})$.

Proposition 2.1.13. $I(a_0; a_1, \cdots, a_n; a_{n+1})$ defines a multi-valued holomorphic function on $\mathcal{D}_n$ via analytic continuation. The multi-valuedness comes from different choices of the integration path.

Proof. According to Lemma 2.1.11, $I(a_0; a_1, \cdots, a_n; a_{n+1})$ has distinct local power series expansions depending on the homotopy class of γ. They define a collection of holomorphic germs Γ which is a locally connected subspace of the etale space of $\mathcal{O}_{\mathcal{D}_n}$. It is not difficult to see that Γ is connected since it is possible to deform between any two integration paths. Therefore, we may think of $I(a_0; a_1, \cdots, a_n; a_{n+1})$ as a maximal analytic continuation of any of its germ (see [20], Chapter 1, Section 7). $\square$

2.1.2 Regularization

In this section, we define the iterated integral (2.1.8) when $a_0 = a_1$ or $a_n = a_{n+1}$, where the integral diverges. This process is known as regularization. This is extensively discussed in [21] and [19]. To carry out this process we need to introduce shuffle algebra and Lyndon words.

The non-commutative polynomial algebra $\mathbb{Q}\langle X_0, X_1, \cdots, X_{d+1}\rangle$ has a shuffle product $ш$ besides product (which is concatenation), defined inductively by $1 ш X_i = X_i ш 1 = X_i$ and

$$\begin{aligned}(X_{i_1}X_{i_2}\cdots X_{i_n}) ш (X_{j_1}X_{j_2}\cdots X_{j_m}) &= X_{i_1}(X_{i_2}\cdots X_{i_n} ш X_{j_1}X_{j_2}\cdots X_{j_m}) \\ &\quad + X_{j_1}(X_{i_1}X_{i_2}\cdots X_{i_n} ш X_{j_2}\cdots X_{j_m}) \qquad (2.1.13)\end{aligned}$$

Multiplication of iterated integrals (2.1.6 iii.) also behaves like a shuffle product. In this section, we highlight a way to connect shuffle algebras with hyperlogarithms.

First we need to introduce the concept of a Lyndon word. Suppose there is a total order by index on the alphabet, i.e. $X_i \prec X_j$ if $i < j$, then there is the usual lexicographical order on the set of words on $\{X_i\}_{i\in\mathbb{N}}$.

Definition 2.1.14. A *Lyndon word* $X_{i_1}\cdots X_{i_n}$ is a word such that

$$X_{i_1}\cdots X_{i_r} \preceq X_{i_{r+1}}\cdots X_{i_n}, \qquad \forall 1 \le r < n \tag{2.1.14}$$

Remark 2.1.15. $\overbrace{X_i\cdots X_i}^{n}$ is a Lyndon word. A Lyndon word consists of more than one letters which ordered as $X_{i_1} \prec \cdots \prec X_{i_n}$ cannot begin with X_{i_n} nor end with X_{i_1}.

Example 2.1.16. $X_0X_3X_2X_1$ is a Lyndon word, while $X_1X_2X_3X_0$ isn't.

The following fact is well-known and we omit its proof.

Theorem 2.1.17. [22] $\mathbb{Q}\langle X_0,\cdots,X_{d+1}\rangle$ forms a commutative ring under $\sqcup\!\sqcup$, and the set of Lyndon words form an algebraically independent generating set. In other words, we can express non Lyndon words as Lyndon words in a unique way.

Example 2.1.18. $X_0X_1X_0 = (X_0X_1) \sqcup\!\sqcup X_0 - 2X_0^2X_1$

For fixed $a_0, a_{i_1},\cdots,a_{i_m},a_{n+1}$ and a path γ from a_0 to a_{n+1}, we may construct a multiplicative map

$$\iota_\gamma : \mathbb{Q}\langle X_0,\cdots,X_{d+1}\rangle \to \mathbb{C}, \quad X_{i_m}\cdots X_{i_1} \mapsto I_\gamma(a_0;a_{i_1},\cdots,a_{i_m};a_{n+1})$$

it is clear $\iota_\gamma(w_1 \sqcup\!\sqcup w_2) = \iota_\gamma(w_1)\iota_\gamma(w_2)$ for any words $w_1, w_2 \in \mathbb{Q}\langle X_0,\cdots,X_{d+1}\rangle$.

Unfortunately, $I_\gamma(a_0;a_{i_1},\cdots,a_{i_m};a_{n+1})$ doesn't make sense in (2.1.2) if $a_{i_1} = a_0$ or $a_{i_m} = a_{n+1}$ since the integral then becomes singular at the end points. A workaround is to consider its regularized value.

Consider a piece-wise smooth path from a_0 to a_{n+1} with $\gamma'(0) = \lambda$, $\gamma'(1) = \mu$. If we denote $\gamma|_{[\varepsilon,1-\varepsilon]}$ as γ_ε for $\varepsilon > 0$ small, we can write $\gamma(\varepsilon) = a_0 + \lambda\varepsilon + \varepsilon f(\varepsilon)$, $\gamma(1-\varepsilon) = a_0 - \mu\varepsilon + \varepsilon g(\varepsilon)$, for some smooth functions f, g such that $\lim\limits_{\varepsilon\to 0} f(\varepsilon) = \lim\limits_{\varepsilon\to 0} g(\varepsilon) = 0$. Then by Example 2.1.4 we have

$$\begin{aligned} I_{\gamma_\varepsilon}(a_0; \overbrace{a_0, \cdots, a_0}^{m}; a_{n+1}) &= \frac{1}{m!}\left(\log(\gamma(1-\varepsilon) - a_0) - \log(\gamma(\varepsilon) - a_0)\right)^m \\ &= \frac{1}{m!}\left(\log(a_{n+1} - a_0 - \mu\varepsilon + \varepsilon g(\varepsilon)) - \log(\lambda\varepsilon + \varepsilon f(\varepsilon))\right)^m \end{aligned} \tag{2.1.15}$$

$$\begin{aligned} I_{\gamma_\varepsilon}(a_0; \overbrace{a_{n+1}, \cdots, a_{n+1}}^{m}; a_{n+1}) &= \frac{1}{m!}\left(\log(\gamma(1-\varepsilon) - a_{n+1}) - \log(\gamma(\varepsilon) - a_{n+1})\right)^m \\ &= \frac{1}{m!}\left(\log(-\mu\varepsilon + \varepsilon g(\varepsilon)) - \log(a_0 - a_{n+1} + \lambda\varepsilon + \varepsilon g(\varepsilon))\right)^m \end{aligned} \tag{2.1.16}$$

Or more generally the following proposition

Proposition 2.1.19. When $\varepsilon > 0$ is small, one has

$$I_{\gamma_\varepsilon}(a_0; a_{i_1}, \cdots, a_{i_m}; a_{n+1}) = f_0(\varepsilon) + f_1(\varepsilon)\log\varepsilon + \cdots + f_m(\varepsilon)\log^m\varepsilon,$$

where f_i are analytic around 0, and $f_0(0)$ is only dependent on the homotopy class of γ with fixed $\gamma'(0)$, $\gamma'(1)$.

Proof. If $i_1 > 0$ and $i_m < n+1$, $f_0(0) = I_\gamma(a_0; a_{i_1}, \cdots, a_{i_m}; a_{n+1})$ will be a regular hyperlogarithm, and $f_1 = \cdots = f_m$ will be all be 0. If either $i_1 = 0$ or $i_m = n+1$, by identifying $I_{\gamma_\varepsilon}(a_0; a_{i_1}, \cdots, a_{i_m}; a_{n+1})$ as $X_{i_1} \cdots X_{i_m}$ and applying Theorem 2.1.17, we may rewrite $I_{\gamma_\varepsilon}(a_0; a_{i_1}, \cdots, a_{i_m}; a_{n+1})$ uniquely as products and sums of regular hyperlogarithms and $I_{\gamma_\varepsilon}(a_0; a_0, \cdots, a_0; a_{n+1})$, $I_{\gamma_\varepsilon}(a_0; a_{n+1}, \cdots, a_{n+1}; a_{n+1})$. $\square$

Definition 2.1.20. The *regularized value* of $I_\gamma(a_0; a_{i_1}, \cdots, a_{i_m}; a_{n+1})$ is defined to be $f_0(0)$.

Therefore, ι_γ is well-defined and multiplicative.

Example 2.1.21. Suppose γ is the straight path from 0 to z. Then the regularized value of $I_\gamma(0;\overbrace{0,\cdots,0}^{n};z)$ is $\dfrac{1}{n!}\log^n z$. The regularized value of $I_\gamma(0;0,a_1,0;z)$ is $I_\gamma(0;a_1,0;z)\log z - 2I_\gamma(0;a_1,0,0;z)$, according to Example 2.1.18.

Remark 2.1.22. Remark 2.1.12 can be generalized for regularized iterated integrals. If either $a_0 = a_0^0$, $\gamma_t'(0) = \gamma_0'(0)$ or $a_{n+1} = a_{n+1}^0$, $\gamma_t'(1) = \gamma_0'(1)$, then γ_1 is homotopic to $\alpha^{-1}\gamma_0\beta$ with fixed $\gamma'(0)$ or $\gamma'(1)$. By Proposition 2.1.19, $I_{\gamma_1}(a_0;a_1,\cdots,a_n;a_{n+1})$, $I_{\alpha^{-1}\gamma_0\beta}(a_0;a_1,\cdots,a_n;a_{n+1})$ have the same regularized values.

2.2 Hopf algebras and Lie coalgebras

Iterated integrals have the structure of a connected graded Hopf algebra. We review the definition and main properties of a connected graded Hopf algebra. They yield a Lie coalgebra when modded out by non-constant products, which form the theoretic basis for the construction of our model for a motivic complex. We also discuss in detail the symbol map $\Delta_{1,\cdots,1}$, which is useful in generating symbols of multiple polylogarithms. The following discussions can be found in many references, for example [23].

2.2.1 Hopf algebra

Definition 2.2.1. Let k be a $\mathbb{Z}$-algebra. A *k-Hopf algebra H* is a k-algebra equipped with k-algebra homomorphisms

- a coproduct $\Delta : H \to H \otimes_k H$

- a counit $\epsilon : H \to k$

- an antipode $S : H \to H$

such that

- Δ is coassociative: $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$

- ϵ is counital: $(1 \otimes \epsilon)\Delta$, $(\epsilon \otimes 1)\Delta$ are the canonical isomorphisms $H \otimes k \cong k \otimes H \cong H$

- $m(S \otimes 1)\Delta = m(1 \otimes S)\Delta = \epsilon$, where $m : H \otimes H \to H$ is multiplication

Definition 2.2.2. A *connected graded Hopf algebra* is a graded k-algebra $\bigoplus_{n\geq 0} H_n$ that is a Hopf algebra and $\Delta(H_n) \subseteq \bigoplus_{p+q=n} H_p \otimes H_q$, $H_0 = k$. We denote the restriction of Δ to H_n as $\Delta_n = \sum_{p+q=n} \Delta_{p,q}$, where $\Delta_{p,q} : H_n \to H_p \otimes H_q$ are the components.

Lemma 2.2.3. For any connected graded Hopf algebra H, $\ker \epsilon = \bigoplus_{n\geq 1} H_n$. If k is a field, then $\ker \epsilon$ is the unique maximal ideal of H.

Proof. Suppose $a \in H_p - \ker \epsilon$ is of the smallest degree such that $p > 0$. Then by county we have

$$a = (\epsilon \otimes 1)\Delta(a) = a + \epsilon(a)$$

This implies $\epsilon(a) = 0$, which is a contradiction. Therefore, $\ker \epsilon = \bigoplus_{n\geq 1} H_n$. In the case where k is a field, since $H - \ker \epsilon = k - \{0\}$ are units, so $\ker \epsilon$ is the unique maximal ideal of H. $\square$

Lemma 2.2.4. $\Delta_{0,n}(a) = 1 \otimes a$ and $\Delta_{n,0}(a) = a \otimes 1$

Proof. Suppose $\Delta_{0,n}(a) = \sum c_i \otimes b_i$, since c_i are scalars, this is equal to $1 \otimes (\sum c_i b_i) = 1 \otimes b$. By Lemma 2.2.3, $a = (\epsilon \otimes 1)\Delta(a) = (\epsilon \otimes 1)\Delta_{0,n}(a) = b$. This justifies $\Delta_{0,n}(a) = 1 \otimes a$. Similar arguments can be made for $\Delta_{n,0}$. $\square$

Definition 2.2.5. We define the restricted coproduct Δ' to be $\bigoplus_{\substack{p+q=n \\ p>0, q>0}} \Delta_{p,q}$.

Due to Lemma 2.2.4, $\Delta'(a) = \Delta(a) - a \otimes 1 - 1 \otimes a$. It is not hard to justify that $(1 \otimes \Delta')\Delta' = (\Delta' \otimes 1)\Delta'$.

Theorem 2.2.6. Suppose H is a graded bialgebra (a Hopf algebra without the antipode) with $H_0 = k$. There exists a unique antipode map S, giving H a Hopf algebra structure.

Proof. First we prove the uniqueness. Suppose S is an antipode, then we have for any $|a| > 0$,

$$\begin{aligned} 0 = \epsilon(a) &= m(1 \otimes S)\Delta(a) \\ &= m(1 \otimes S)(1 \otimes a + a \otimes 1 + \Delta'(a)) \qquad (2.2.1) \\ &= S(a) + a + m(1 \otimes S)\Delta'(a) \end{aligned}$$

This uniquely and inductively defines S as $S(a) = -a - m(1 \otimes S)\Delta'(a)$, and in particular, $S(a) = -a$ for $|a| = 1$. For the existence of antipodes, we simply define inductively that $S(a) = -a - m(1 \otimes S)\Delta'(a)$. We still need to check that $m(1 \otimes S)\Delta' = m(S \otimes 1)\Delta'$. Suppose this is true for all $|a| < n$, then for $|a| = n$, assume $\Delta(a) = \sum_i a_{i_1} \otimes a_{i_2}$, we have

$$\begin{aligned} m(1 \otimes S)\Delta' &= m\left(1 \otimes (-\operatorname{id} - m(1 \otimes S)\Delta')\right)\Delta' \\ &= -m\Delta' - \left(1 \otimes m(S \otimes 1)\Delta'\right)\Delta' \\ &= -m\Delta' - m(1 \otimes S \otimes 1)(1 \otimes \Delta')\Delta' \\ &= -m\Delta' - m(1 \otimes S \otimes 1)(\Delta' \otimes 1)\Delta' \\ &= -m\Delta' - \left(m(1 \otimes S)\Delta' \otimes 1\right)\Delta' \\ &= m\left((-\operatorname{id} - m(S \otimes 1)\Delta') \otimes 1\right)\Delta' \\ &= m(S \otimes 1)\Delta' \end{aligned} \tag{2.2.2}$$

□

Definition 2.2.7. We say an ideal $I \subseteq H$ is a Hopf ideal if $\Delta(I) \subseteq I \otimes H + H \otimes I$ and $S(I) \subseteq I$.

Proposition 2.2.8. If $I \subseteq$ is a Hopf ideal, then the quotient algebra H/I is again a Hopf algebra. If H is graded and I is homogeneous, H/I is also graded.

Proof. Due to the definition of Hopf ideal, the induced coproduct Δ and induced antipode S are well-defined. So the quotient algebra is a Hopf algebra as well. □

Definition 2.2.9. Suppose A is an abelian group, consider the tensor algebra $T(A) = \bigoplus_{n=0}^{\infty} A^{\otimes n}$, endowed with a shuffle product

$$\begin{gathered} (a \otimes w_1) \sqcup\!\sqcup (b \otimes w_2) = a \otimes (w_1 \sqcup\!\sqcup (b \otimes w_2)) + b \otimes ((a \otimes w_1) \sqcup\!\sqcup w_2) \\ \forall a, b \in A, w_1 \in A^{\otimes n}, w_2 \in A^{\otimes m} \end{gathered} \tag{2.2.3}$$

The following results are well-known, yet their proofs are frequently omitted. We present them here for completeness and future reference.

Proposition 2.2.10. We can define a graded connected Hopf structure over this shuffle algebra, with restricted coproduct

$$\Delta'(a_1 \otimes \cdots \otimes a_n) = \sum_{i=1}^{n-1} (a_1 \otimes \cdots \otimes a_i) \bigotimes (a_{i+1} \otimes \cdots \otimes a_n) \tag{2.2.4}$$

and antipode

$$S(a_1 \otimes \cdots \otimes a_n) = (-1)^n a_n \otimes \cdots \otimes a_1 \tag{2.2.5}$$

Proof. Note that coproduct is defined like deconcatenations of a word. It is trivial to check coassociativity and counity. We are left to check that the antipode satisfies

$$S(a_1 \otimes \cdots \otimes a_n) + a_1 \otimes \cdots \otimes a_n + m(1 \otimes S)\Delta'(a_1 \otimes \cdots \otimes a_n) = 0 \tag{2.2.6}$$

If $n = 2$, this is trivial, and for $n \geq 3$ we see that $m(1 \otimes S)\Delta'(a_1 \otimes \cdots \otimes a_n)$ equals

$$\begin{aligned}
&\sum_{i=1}^{n-1}(-1)^i a_i \otimes \cdots \otimes a_1 \sqcup a_{i+1} \otimes \cdots \otimes a_n \\
&= \sum_{i=1}^{n-1} \Big((-1)^i a_i \otimes (a_{i-1} \otimes \cdots \otimes a_1 \sqcup a_{i+1} \otimes \cdots \otimes a_n) \\
&\qquad + (-1)^i a_{i+1} \otimes (a_i \otimes \cdots \otimes a_1 \sqcup a_{i+2} \otimes \cdots \otimes a_n)\Big) \\
&= \sum_{i=1}^{n-1}(-1)^i a_i \otimes (a_{i-1} \otimes \cdots \otimes a_1 \sqcup a_{i+1} \otimes \cdots \otimes a_n) \\
&\quad - \sum_{i=2}^{n}(-1)^i a_i \otimes (a_{i-1} \otimes \cdots \otimes a_1 \sqcup a_{i+1} \otimes \cdots \otimes a_n) \\
&= -a_1 \otimes a_2 \otimes \cdots \otimes a_n - (-1)^n a_n \otimes a_{n-1} \otimes \cdots \otimes a_1
\end{aligned}$$

This proves (2.2.6). □

Definition 2.2.11. Due to coassociativity, we can decompose $\Delta^{\circ k}$ into the sum of components of

$$\Delta_{n_1,\cdots,n_k} : H_n \to \bigoplus_{\substack{n_1+\cdots+n_k=n \\ n_i \geq 0}} H_{n_1} \otimes \cdots \otimes H_{n_k}$$

And we denote

$$\Delta_{n_1,\cdots,n_k}(a) = \sum_i a_{i1}^{(n_1,\cdots,n_k)} \otimes \cdots \otimes a_{ik}^{(n_1,\cdots,n_k)} \tag{2.2.7}$$

We omit $(n_1, \cdots, n_k)$ when the context is clear.

Definition 2.2.12. The symbol map $\Delta_{1,\cdots,1} : H \to T^*H_1$ is the direct sum of $\Delta_{1,\cdots,1} : H_n \to T^nH_1$

Proposition 2.2.13. $\Delta_{1,\cdots,1}$ is a morphism of graded Hopf algebras.

Proof. To show that $\Delta_{1,\cdots,1}$ preserves product we need

$$\Delta_{1,\cdots,1}(ab) = \Delta_{1,\cdots,1}(a) \sqcup \Delta_{1,\cdots,1}(b) \tag{2.2.8}$$

for any $|a| = k, |b| = l$. Let's assume

$$\Delta_{1,k-1}(a) = \sum_i a_{i1}^{(1,k-1)} \otimes a_{i2}^{(1,k-1)}, \qquad \Delta_{1,l-1}(b) = \sum_j b_{j1}^{(1,l-1)} \otimes b_{j2}^{(1,l-1)}.$$

Since $\Delta(ab) = \Delta(a)\Delta(b)$, we have

$$\begin{aligned}
\Delta_{1,k+l-1}(ab) &= \Delta_{1,k-1}(a)\Delta_{0,l}(b) + \Delta_{0,k}(a)\Delta_{1,l-1}(b) \\
&= \left(\sum_i a_{i1} \otimes a_{i2}\right)(1 \otimes b) + (1 \otimes a)\left(\sum_j b_{j1} \otimes b_{j2}\right) \\
&= \sum_i a_{i1} \otimes (a_{i2}b) + \sum_j b_{j1} \otimes (ab_{j2}),
\end{aligned}$$

Note that

$$\Delta_{1,\cdots,1}(a) = (\mathrm{id} \otimes \Delta_{1,\cdots,1}) \circ \Delta_{1,k-1}(a) = \sum_i a_{i1} \otimes \Delta_{1,\cdots,1}(a_{i2}),$$

similarly

$$\Delta_{1,\cdots,1}(b) = (\mathrm{id} \otimes \Delta_{1,\cdots,1}) \circ \Delta_{1,l-1}(b) = \sum_j b_{j1} \otimes \Delta_{1,\cdots,1}(b_{j2}).$$

So we have

$$\begin{aligned}
\Delta_{1,\cdots,1}(ab) &= (\mathrm{id} \otimes \Delta_{1,\cdots,1}) \circ \Delta_{1,k+l-1}(ab) \\
&= \sum_i a_{i1} \otimes \Delta_{1,\cdots,1}(a_{i2}b) + \sum_j b_{j1} \otimes \Delta_{1,\cdots,1}(ab_{j2}) \\
&= \sum_i a_{i1} \otimes (\Delta_{1,\cdots,1}(a_{i2}) \sqcup\!\sqcup \Delta_{1,\cdots,1}(b)) + \sum_j b_{j1} \otimes (\Delta_{1,\cdots,1}(a) \sqcup\!\sqcup \Delta_{1,\cdots,1}(b_{j2}))
\end{aligned}$$

The last equality is by induction on lower weight ((2.2.8) is automatic when $k = 0$).

On the other hand

$$\begin{aligned}
&\Delta_{1,\cdots,1}(a) \sqcup\!\sqcup \Delta_{1,\cdots,1}(b) \\
&= \left(\sum_i a_{i1} \otimes \Delta_{1,\cdots,1}(a_{i2})\right) \sqcup\!\sqcup \left(\sum_j b_{j1} \otimes \Delta_{1,\cdots,1}(b_{j2})\right) \\
&= \sum_{i,j} a_{i1} \otimes \Delta_{1,\cdots,1}(a_{i2}) \sqcup\!\sqcup b_{j1} \otimes \Delta_{1,\cdots,1}(b_{j2}) \\
&= \sum_{i,j} a_{i1} \otimes (\Delta_{1,\cdots,1}(a_{i2}) \sqcup\!\sqcup b_{j1} \otimes \Delta_{1,\cdots,1}(b_{j2})) + b_{j1} \otimes (a_{i1} \otimes \Delta_{1,\cdots,1}(a_{i2}) \sqcup\!\sqcup \Delta_{1,\cdots,1}(b_{j2})) \\
&= \sum_i a_{i1} \otimes \left(\Delta_{1,\cdots,1}(a_{i2}) \sqcup\!\sqcup \sum_j b_{j1} \otimes \Delta_{1,\cdots,1}(b_{j2})\right) \\
&\quad + \sum_j b_{j1} \otimes \left(\sum_i a_{i1} \otimes \Delta_{1,\cdots,1}(a_{i2}) \sqcup\!\sqcup \Delta_{1,\cdots,1}(b_{j2})\right) \\
&= \sum_i a_{i1} \otimes (\Delta_{1,\cdots,1}(a_{i2}) \sqcup\!\sqcup \Delta_{1,\cdots,1}(b)) + \sum_j b_{j1} \otimes (\Delta_{1,\cdots,1}(a) \sqcup\!\sqcup \Delta_{1,\cdots,1}(b_{j2}))
\end{aligned}$$

To show $\Delta_{1,\cdots,1}$ preserves coproduct (to reduce confusion, we use $\bigotimes$ in $T(A) \bigotimes T(A)$) we need

$$(\Delta_{1,\cdots,1} \bigotimes \Delta_{1,\cdots,1}) \circ \Delta(a) = \Delta \circ \Delta_{1,\cdots,1}(a) \tag{2.2.9}$$

First note that for any $|a| = n$

$$\begin{aligned}
\Delta_{1,\cdots,1}(a) &= (\Delta_{1,\cdots,1} \otimes \Delta_{1,\cdots,1}) \circ \Delta_{k,n-k}(a) \\
&= \Delta_{1,\cdots,1} \otimes \Delta_{1,\cdots,1} \left(\sum_{i_k} a_{i_k 1}^{(k,n-k)} \otimes a_{i_k 2}^{(k,n-k)} \right) \\
&= \sum_{i_k} \left(\Delta_{1,\cdots,1} \left(a_{i_k 1}^{(k,n-k)} \right) \otimes \Delta_{1,\cdots,1} \left(a_{i_k 2}^{(k,n-k)} \right) \right) \\
&= \sum_{j} a_{j1}^{(1,\cdots,1)} \otimes \cdots \otimes a_{jn}^{(1,\cdots,1)}
\end{aligned}$$

Therefore

$$\begin{aligned}
(\Delta_{1,\cdots,1} \bigotimes \Delta_{1,\cdots,1}) \circ \Delta(a) &= \sum_{k=0}^{n} \sum_{i_k} \Delta_{1,\cdots,1} \left(a_{i1}^{(k,n-k)} \right) \bigotimes \Delta_{1,\cdots,1} \left(a_{i2}^{(k,n-k)} \right) \\
&= \sum_{k=0}^{n} \sum_{j} a_{j1}^{(1,\cdots,1)} \otimes \cdots \otimes a_{jk}^{(1,\cdots,1)} \bigotimes a_{j(k+1)}^{(1,\cdots,1)} \otimes \cdots \otimes a_{jn}^{(1,\cdots,1)} \\
&= \sum_{j} \sum_{k=0}^{n} a_{j1}^{(1,\cdots,1)} \otimes \cdots \otimes a_{jk}^{(1,\cdots,1)} \bigotimes a_{j(k+1)}^{(1,\cdots,1)} \otimes \cdots \otimes a_{jn}^{(1,\cdots,1)} \\
&= \sum_{j} \Delta \left(a_{j1}^{(1,\cdots,1)} \otimes \cdots \otimes a_{jn}^{(1,\cdots,1)} \right) \\
&= \Delta \left(\sum_{j} a_{j1}^{(1,\cdots,1)} \otimes \cdots \otimes a_{jn}^{(1,\cdots,1)} \right) \\
&= \Delta \circ \Delta_{1,\cdots,1}(a)
\end{aligned}$$

$\square$

2.2.2 Lie coalgebra

The quotient of a Hopf algebra H by products admits the structure of a Lie coalgebra, which is graded if H is connected graded. We now give the formal definition of a Lie coalgebra.

Definition 2.2.14. Let k be a $\mathbb{Z}$-algebra. A Lie coalgebra L is a module over k with a linear mapping called the *cobracket* $\delta : L \to L \wedge L$, such that $(\delta a) \wedge b = a \wedge (\delta b)$ and $\delta(1 \wedge \delta) = \delta(\delta \wedge 1)$. δ can be extend to a linear mapping $\delta : \bigwedge^n L \to \bigwedge^{n+1} L$

$$\delta(a_1 \wedge \cdots \wedge a_n) = \sum_{i=1}^{n} (-1)^{i+1} a_1 \wedge \cdots \wedge (\delta a_i) \wedge \cdots \wedge a_n \tag{2.2.10}$$

This forms a chain complex known as the *Chevalley-Eilenberg complex* $\bigwedge^* L$

$$L \xrightarrow{\delta} L \wedge L \xrightarrow{\delta} L \wedge L \wedge L \xrightarrow{\delta} \cdots \tag{2.2.11}$$

Definition 2.2.15. We say that a Lie coalgebra L is *graded* if $L = \bigoplus_i L_i$ and $\delta(L_n) \subseteq \bigoplus_{p+q=n} L_p \wedge L_q$. The degree n part of $\bigwedge^* L$ is

$$L_n \to \bigoplus_{p+q=n} L_p \wedge L_q \to \bigoplus_{p+q+r=n} L_p \wedge L_q \wedge L_r \to \cdots \to L_1^{\wedge n}$$

denoted as $(\bigwedge^* L)_n$

Example 2.2.16. $(\bigwedge^* L)_2$ reads

$$L_2 \to \bigwedge^2 L_1$$

$(\bigwedge^* L)_3$ reads

$$L_3 \to L_2 \otimes L_1 \to \bigwedge^3 L_1$$

$(\bigwedge^* L)_4$ reads

$$L_4 \to L_3 \otimes L_1 \oplus \bigwedge^2 L_2 \to L_2 \oplus \bigwedge^2 L_1 \to \bigwedge^4 L_1$$

It is known that there exists a projection map on a connected graded Hopf algebra H over $\mathbb{Q}$, whose image can be canonically identified with H modulo products.

Definition 2.2.17. [13] Suppose H is a connected graded Hopf algebra over a field $\mathbb{Q}$ and denote $H_{>0} = \bigoplus_{i>0} H_i$. First we define $R : H \to H$ inductively by $R(x) = nx - m(1 \otimes R)\Delta'(x)$ where $|x| = n$ and m is multiplication. The projection map $P : H \to H$ is simply such that $P|_{H_n} = \dfrac{1}{n} R_n$.

Proposition 2.2.18. *P is an idempotent map with $\ker P = H_{>0} \cdot H_{>0}$.*

Proof. It is straightforward to check that $\ker P \subseteq H_{>0} \cdot H_{>0}$. Let's prove $H_{>0} \cdot H_{>0} \subseteq \ker P$. Suppose $|x| = k$, $|y| = l$ and $k, l > 0$, and write

$$\Delta'(x) = \sum_i x_{i1} \otimes x_{i2}, \qquad \Delta'(y) = \sum_j y_{j1} \otimes y_{j2}$$

Then we have

$$R(x) = kx - \sum_i x_{i1} R(x_{i2}), \qquad R(y) = ly - \sum_j y_{j1} R(y_{j2})$$

and

$$\begin{aligned}\Delta'(xy) = \sum_{i,j} x_{i1} y_{j1} \otimes x_{i2} y_{j2} + \sum_i (x_{i1} y \otimes x_{i2} + x_{i1} \otimes x_{i2} y) \\ + \sum_j (x y_{j1} \otimes y_{j2} + y_{j1} \otimes x y_{j2}) + x \otimes y + y \otimes x\end{aligned}$$

By induction, we may assume that R annihilates products of lower weights, and we get

$$\begin{aligned}R(xy) &= (k+l)xy - m(1 \otimes R)\Delta'(xy) \\ &= (k+l)xy - \left(\sum_i x_{i1} y R(x_{i2}) + \sum_j x y_{j1} R(y_{j2}) + xR(y) + yR(x) \right) \\ &= (k+l)xy - y \left(\sum_i x_{i1} R(x_{i2}) + R(x) \right) - x \left(\sum_j y_{j1} R(y_{j2}) + R(y) \right) \\ &= (k+l)xy - kxy - lxy = 0\end{aligned}$$

P is an idempotent because

$$P^2(x) = \frac{1}{k} R \left(x - \frac{1}{k} \sum_i x_{i1} R(x_{i2}) \right) = \frac{1}{k} R(x) - \frac{1}{k} \sum_i R\left(x_{i1} R(x_{i2})\right) = P(x)$$

□

Therefore $H = \operatorname{im} P \oplus \ker P$. We call $\operatorname{im} P$ the subspace of indecomposable elements.

Theorem 2.2.19. *H modulo products has a Lie coalgebra structure, and there is an isomorphism*

$$\frac{H}{H_{>0} \cdot H_{>0}} \cong \operatorname{im} P, \qquad \bar{x} \mapsto P(x)$$

with cobracket $(P \wedge P)\Delta : \operatorname{im} P \to \operatorname{im} P \wedge \operatorname{im} P$

Proof. We only need the following diagram, which commutes because $(P \wedge P)\Delta P = (P \wedge P)\Delta$.

$$\begin{tikzcd}
\overline{H} \arrow[r, "P"] \arrow[d, "\delta"'] & \operatorname{im} P \arrow[d, "(P\wedge P)\Delta"] \\
\overline{H} \wedge \overline{H} \arrow[r, "P\wedge P"] & \operatorname{im} P \wedge \operatorname{im} P
\end{tikzcd}$$

□

The following is elementary.

Proposition 2.2.20. *If $\phi : H \to H'$ is a morphism between connected graded Hopf algebras, then it induces a morphism between the Lie coalgebras $H/(H_{>0} \cdot H_{>0}) \to H'/(H'_{>0} \cdot H'_{>0})$.*

2.3 Multiple polylogarithms

Multiple polylogarithms are multivariate multi-valued holomorphic functions that are often defined as the analytic continuation of some power series. However, it is more convenient to define them as iterated integrals (Lemma 2.1.9).

2.3.1 Multiple polylogarithms

Multiple polylogarithms naturally live on $\mathbb{C}^d - |D|$ for some simple normal crossing divisor D. Its multi-valuedness comes from looping around the divisors. First we give a clear description of this natural domain

Definition 2.3.1.

$$S_d(\mathbb{C}) = \left\{ (x_1, \ldots, x_d) \in \mathbb{C}^d \middle| x_i \neq 0, \prod_{r=j}^{k} x_r \neq 1, \forall 1 \leq j \leq k \leq d \right\}. \tag{2.3.1}$$

We also write $\widetilde{S}_d(\mathbb{C}) \xrightarrow{\widetilde{\pi}} S_d(\mathbb{C})$ and $\widehat{S}_d(\mathbb{C}) \xrightarrow{\widehat{\pi}} S_d(\mathbb{C})$ as the universal cover and the universal abelian cover of $S_d(\mathbb{C})$. It is clear that $\mathrm{Li}_{n_1,\cdots,n_d}(x_1, \cdots, x_d)$ can be analytically continued as $\mathrm{Li}_{n_1,\cdots,n_d}(\tilde{x}_1, \cdots, \tilde{x}_d)$ throughout all of $\widetilde{S}_d(\mathbb{C})$. And for convenience, we also give a concrete model for $\widehat{S}_d(\mathbb{C})$ ([2], for more details see [15])

$$\widehat{S}_d(\mathbb{C}) = \left\{ (u_i, v_{j,k}) \in \mathbb{C}^{d+\binom{d+1}{2}} \middle| \exp\left(\sum_{r=j}^{k} u_i\right) + \exp(v_{j,k}) = 1, \forall 1 \leq j \leq k \leq d \right\}. \tag{2.3.2}$$

Here u_i, $v_{j,k}$ should be thought of as $\log x_i$, $\log(1 - x_j \cdots x_k)$. In other words, the coordinates on $\widehat{S}_d(\mathbb{C})$ are logarithms of x_i and $1 - x_j \cdots x_k$. It is often convenient to write v_j in place of $v_{j,j}$, and we shall use this convention throughout the thesis. When $d = 1$, it simply becomes $\widehat{\mathbb{C}}$ defined in (1.1.4).

Example 2.3.2.

$$S_2(\mathbb{C}) = \{(x_1, x_2) \in \mathbb{C}^2 | x_1x_2 \neq 0, x_1 \neq 1, x_2 \neq 1, x_1x_2 \neq 1\}$$

$$\widehat{S}_2(\mathbb{C}) = \left\{ (u_1, u_2, v_1, v_2, v_{1,2}) \in \mathbb{C}^5 \middle| \substack{\exp(u_1)+\exp(v_1)=1 \\ \exp(u_2)+\exp(v_2)=1 \\ \exp(u_1+u_2)+\exp(v_{1,2})=1} \right\}$$

Definition 2.3.3. Suppose $\gamma : [0,1] \to \mathbb{C}$ is the straight path from 0 to 1 with $\gamma(t) = t$, and $|x_i| < 1$. We could define the multiple polylogarithm of depth d with weights $n_1, \cdots, n_d$ as

$$\mathrm{Li}_{n_1,\cdots,n_d}(x_1, \cdots, x_d) = (-1)^d I_\gamma(0; (x_1 \cdots x_d)^{-1}, \overbrace{0, \cdots, 0}^{n_1-1}, \cdots, x_d^{-1}, \overbrace{0, \cdots, 0}^{n_d-1}; 1) \tag{2.3.3}$$

We also refer to $n_1 + \cdots + n_d$ as the *weight (or total weight)*. When we replace a_i with $(x_i \cdots x_d)^{-1}$ in (2.1.9) we get the power series expansion of the multiple polylogarithm

$$\sum_{1 \leq m_1 < \cdots < m_d} \frac{x_1^{n_1} \cdots x_d^{n_d}}{m_1^{n_1} \cdots m_d^{n_d}} \tag{2.3.4}$$

By differentiating each term one can compute the derivatives.

Proposition 2.3.4. If $n_i \geq 2$, we have

$$\frac{\partial}{\partial x_i} \mathrm{Li}_{n_1,\cdots,n_d}(x_1, \cdots, x_d) = \frac{1}{x_i} \mathrm{Li}_{n_1,\cdots,n_i-1,\cdots,n_d}(x_1, \cdots, x_d) \tag{2.3.5}$$

If $n_i = 1$, we have three different cases

$$\frac{\partial}{\partial x_d} \mathrm{Li}_{n_1,\cdots,1}(x_1, \cdots, x_d) = \frac{1}{1 - x_d} \mathrm{Li}_{n_1,\cdots,n_{d-1}}(x_1, \cdots, x_{d-1}x_d) \tag{2.3.6}$$

$$\begin{aligned} \frac{\partial}{\partial x_1} \mathrm{Li}_{1,\cdots,n_d}(x_1, \cdots, x_n) &= \frac{1}{1 - x_1} \mathrm{Li}_{n_2,\cdots,n_d}(x_2, \cdots, x_d) \\ &- \frac{1}{1 - x_1} \mathrm{Li}_{n_2,\cdots,n_d}(x_1x_2, \cdots, x_d) - \frac{1}{x_1} \mathrm{Li}_{n_2,\cdots,n_d}(x_1x_2, \cdots, x_d) \end{aligned} \tag{2.3.7}$$

$$\frac{\partial}{\partial x_i} \mathrm{Li}_{n_1,\cdots,1,\cdots,n_d}(x_1,\cdots,x_d) = \frac{1}{1-x_i} \mathrm{Li}_{n_1,\cdots,\widehat{1},\cdots,n_d}(x_1,\cdots,x_{i-1}x_i,\cdots,x_d)$$
$$-\frac{1}{1-x_i} \mathrm{Li}_{n_1,\cdots,\widehat{1},\cdots,n_d}(x_1,\cdots,x_ix_{i+1},\cdots,x_d) - \frac{1}{x_i} \mathrm{Li}_{n_1,\cdots,\widehat{1},\cdots,n_d}(x_1,\cdots,x_ix_{i+1},\cdots,x_d) \tag{2.3.8}$$

Example 2.3.5.

$$\frac{\partial}{\partial x_1} \mathrm{Li}_{2,1,1}(x_1,x_2,x_3) = \frac{1}{x_1} \mathrm{Li}_{1,1,1}(x_1,x_2,x_3)$$

$$\frac{\partial}{\partial x_2} \mathrm{Li}_{2,1,1}(x_1,x_2,x_3) = \frac{1}{1-x_2} \mathrm{Li}_{2,1}(x_1x_2,x_3) - \frac{1}{1-x_2} \mathrm{Li}_{2,1}(x_1,x_2x_3) - \frac{1}{x_2} \mathrm{Li}_{2,1}(x_1,x_2x_3)$$

$$\frac{\partial}{\partial x_3} \mathrm{Li}_{2,1,1}(x_1,x_2,x_3) = \frac{1}{1-x_3} \mathrm{Li}_{2,1}(x_1,x_2)$$

2.3.2 Goncharov's inversion formula for multiple polylogarithms

In [21], Goncharov gives an inductive inversion formula for multiple polylogarithms. His formulation involves Bernoulli numbers and Bernoulli polynomials, so let us first recall these definitions.

Definition 2.3.6. The *Bernoulli numbers* B_n are the coefficients of the formal series $\frac{t}{e^t-1} = \sum_{n\geq 0} \frac{\mathsf{B}_n t^n}{n!}$. The n-th *Bernoulli polynomial* is $B_n(x) = \sum_{k=0}^{n} \binom{n}{k} \mathsf{B}_{n-k} x^k$.

Theorem 2.3.7. [21] Suppose $(x_1,\cdots,x_d) \in S_d(\mathbb{C})$ with $|x_i| = 1, \forall 1 \leq i \leq d$,

$\log(-1) = \pi i$. We have

$$\mathrm{Li}_{n_d,\cdots,n_1}(x_d^{-1},\cdots,x_1^{-1}) = -\sum_{r=0}^{d-1}\mathrm{Li}_{n_r,\cdots,n_1}(x_r^{-1},\cdots,x_1^{-1})\,\mathrm{Li}_{n_{r+1},\cdots,n_d}(x_{r+1},\cdots,x_d) -$$
$$\sum_{r=1}^{d}\sum_{m_1+\cdots+m_d=n_r}(-1)^{1+n_r+\cdots+n_d+m_{r+1}+\cdots+m_d}\prod_{\substack{1\le i\le d\\ i\neq r}}\binom{n_i+m_i-1}{n_i-1}B_{m_r}\left(\frac{\log(x_1\cdots x_d)}{2\pi i}\right)$$
$$\frac{(2\pi i)^{m_r}}{m_r!}\mathrm{Li}_{n_{r-1}+m_{r-1},\cdots,n_1+m_1}(x_{r-1}^{-1},\cdots,x_1^{-1})\,\mathrm{Li}_{n_{r+1}+m_{r+1},\cdots,n_d+m_d}(x_{r+1},\cdots,x_d) \tag{2.3.9}$$

Example 2.3.8.

$$\mathrm{Li}_n(x) + (-1)^n\,\mathrm{Li}_n(x^{-1}) = -\frac{(2\pi i)^n}{n!}B_n\left(\frac{\log x}{2\pi i}\right) \tag{2.3.10}$$

$$\begin{aligned}\mathrm{Li}_{1,1}(x_2^{-1},x_1^{-1}) = &-\mathrm{Li}_{1,1}(x_1,x_2) + \mathrm{Li}_1(x_1)\,\mathrm{Li}_1(x_2) - \mathrm{Li}_2(x_1) + \mathrm{Li}_2(x_2)\\ &+ \mathrm{Li}_1(x_2)\log(x_1) + \mathrm{Li}_1(x_1)\log(x_1x_2) - \mathrm{Li}_1(x_2)\log(x_1x_2)\\ &-\frac{1}{2}\log^2(x_1) + \log(x_1x_2)\log(x_1) - i\pi\log(x_1x_2) - i\pi\,\mathrm{Li}_1(x_1) - \frac{2\pi^2}{3}\end{aligned} \tag{2.3.11}$$

2.3.3 Coproduct on the Hopf algebra of iterated integals

In this section, we discuss a coproduct defined on the Hopf algebra of iterated intergrals. To motivate this, we first need to talk about a coproduct on the Hopf algebra of rooted trees.

Theorem 2.3.9 (Connes-Kreimer's coproduct on Hopf algebra of rooted trees)**.** Consider the free graded algebra generated by rooted trees $\mathcal{T}$. The weight of a tree is the number of leaves. It has a Hopf algebra structure with coproduct of a tree T being

$$\Delta(T) = T\otimes 1 + 1\otimes T + \sum_{\text{admissible cuts}} R(T)\otimes P(T) \tag{2.3.12}$$

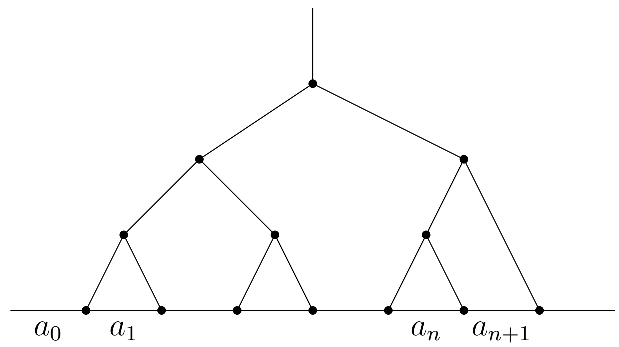


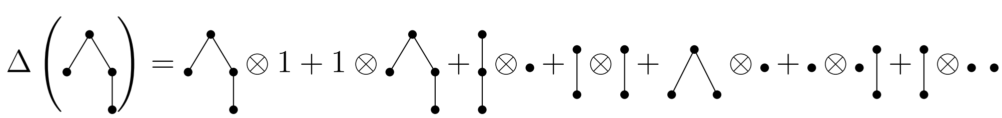
Figure 2.3.1: Decoration on a rooted plane trivalent tree

Here an admissible cut of the tree T is to remove edges such that each path from the root to any leaf sees only one removed edge. After an admissible cut, T becomes of set of rooted trees. $R(T)$ is the one which contains the root, and $P(T)$ is the product of the other ones.

Remark 2.3.10. This is slightly different than in the literature, where normally the Connes-Kreimer's coproduct is defined to be

$$\Delta(T) = T \otimes 1 + 1 \otimes T + \sum_{\text{admissible cuts}} P(T) \otimes R(T)$$

Example 2.3.11.



Definition 2.3.12. (see [8], section 4) Suppose S is a set. An S-decoration of a rooted plane trivalent tree T with $n+1$ leaves is a tuple $(a_0; a_1, \cdots, a_n; a_{n+1}) \in S^{n+2}$. See Figure 2.3.1 for an illustration.

We could think of a rooted trivalent plane tree growing down from the root (root is the only node that has one distinguished leg), and all leaves lie on an imaginary line which has been divided by leaves into $n+2$ parts labeled as $a_0, a_1, \cdots, a_n, a_{n+1}$ in order.

Remark 2.3.13. A subtree of T also has a decoration from the decoration of T, which is just a contiguous subsequence $(a_i; a_{i+1}, \cdots, a_{j-1}; a_j)$.

Theorem 2.3.14. (see [8], section 4) The free algebra $\mathcal{T}(S)$ generated by S-decorated rooted plane trivalent trees again forms a graded Hopf algebra with Connes-Kreimer's coproduct on trees.

Theorem 2.3.15. (see [8], section 4) Suppose S is a set, and let $\widetilde{\mathcal{I}}(S)$ be the free graded algebra generated by $I(a_0; a_1, \cdots, a_n; a_{n+1}), a_i \in S$ in weight $n \geq 1$. Then it is a graded Hopf algebra with coproduct

$$\Delta I(a_0; a_1, \cdots; a_n; a_{n+1}) = \sum_{0=i_0<i_1<\cdots<i_k<i_{k+1}=n+1} I(a_{i_0}; a_{i_1}, \cdots, a_{i_k}; a_{i_{k+1}}) \otimes \prod_{p=0}^{k} I(a_{i_p}; a_{i_p+1}, \cdots, a_{i_{p+1}-1}; a_{i_{p+1}}) \tag{2.3.13}$$

Here we assume $I(a; b) = 1$ for all a, b.

Proof. There is an embedding of $\widetilde{\mathcal{I}}(S)$ into $\mathcal{T}(S)$ that maps $I(a_0; a_1, \cdots, a_n; a_{n+1})$ to the sum of rooted plane trivalent trees with decoration $(a_0; a_1, \cdots, a_n; a_{n+1})$. $\square$

Example 2.3.16.

$$\begin{aligned}\Delta I(a_0; a_1, a_2, a_3; a_4) = {} & 1 \otimes I(a_0; a_1, a_2, a_3; a_4) + I(a_0; a_1; a_4) \otimes I(a_1; a_2, a_3; a_4) \\ & + I(a_0; a_2; a_4) \otimes I(a_0; a_1; a_2) I(a_2; a_3; a_4) + I(a_0; a_3; a_4) \otimes I(a_0; a_1, a_2; a_3) \\ & + I(a_0; a_1, a_2; a_4) \otimes I(a_2; a_3; a_4) + I(a_0; a_1, a_3; a_4) \otimes I(a_1; a_2; a_3) \\ & + I(a_0; a_2, a_3; a_4) \otimes I(a_0; a_1; a_2) + I(a_0; a_1, a_2, a_3; a_4) \otimes 1\end{aligned}$$

Since an iterated integral evaluates to zero if its integration path is the trivial constant path at a point, we sometimes wish to get rid of such elements in $\widetilde{I}(S)$.

Definition 2.3.17. Elements like $I(a; \cdots; a)$ that start and end with the same element are called *degenerate*. Define $\mathcal{I}(S)$ to be $\widetilde{\mathcal{I}}(S)$ modulo degenerates.

Proposition 2.3.18. $\mathcal{I}(S)$ is a graded Hopf algebra with a coproduct induced from Theorem 2.3.15.

Proof. It is not hard to see that the ideal generated by degenerates is a homogeneous Hopf ideal. So the quotient algebra $\mathcal{I}(S)$ of $\widetilde{\mathcal{I}}(S)$ is again a graded Hopf algebra according to Proposition 2.2.8. □

2.3.4 Symbols of multiple polylogarithms

Herbert Gangl, Goncharov et al. explored the symbols of multiple polylogarithms (see [24]).

Definition 2.3.19. The symbol of a hyperlogarithm $I(a_0; \cdots; a_{n+1})$ is its image under $\Delta_{1,\cdots,1}$

Example 2.3.20. The symbol of $I(a_0; a_1, a_2, a_3; a_4)$ is

$$\begin{aligned}
&I(a_0;a_1;a_4)\otimes I(a_1;a_2;a_4)\otimes I(a_2;a_3;a_4)+I(a_0;a_1;a_4)\otimes I(a_1;a_3;a_4)\otimes I(a_1;a_2;a_3)\\
&+I(a_0;a_2;a_4)\otimes I(a_0;a_1;a_2)\otimes I(a_2;a_3;a_4)+I(a_0;a_2;a_4)\otimes I(a_2;a_3;a_4)\otimes I(a_0;a_1;a_2)\\
&+I(a_0;a_3;a_4)\otimes I(a_0;a_1;a_3)\otimes I(a_1;a_2;a_3)+I(a_0;a_3;a_4)\otimes I(a_0;a_2;a_3)\otimes I(a_0;a_1;a_2)
\end{aligned} \tag{2.3.14}$$

Example 2.3.21. Similarly, we can compute the symbol of multiple polylogarithm $\mathrm{Li}_{2,1}(x_1,x_2)$, the previous example becomes

$$\begin{aligned}\Delta_{1,\cdots,1}(\mathrm{Li}_{2,1}(x_1,x_2)) = \Delta_{1,\cdots,1}\left(I\left(0;\frac{1}{x_1x_2},0,\frac{1}{x_2};1\right)\right) = \\ \mathrm{Li}_1(x_2)\otimes\mathrm{Li}_1(x_1)\otimes\log(x_1) - \mathrm{Li}_1(x_1x_2)\otimes\log(x_1)\otimes\log(x_1) \\ + \mathrm{Li}_1(x_1x_2)\otimes\log(x_1x_2)\otimes\mathrm{Li}_1(x_2) - \mathrm{Li}_1(x_1x_2)\otimes\mathrm{Li}_1(x_1)\otimes\log(x_1) \\ + \mathrm{Li}_1(x_1x_2)\otimes\mathrm{Li}_1(x_2)\otimes\log(x_1)\end{aligned} \quad (2.3.15)$$

2.3.5 Monodromies of multiple polylogarithms

In [9], Zhao described a variation of mixed Hodge structures encoded by multiple polylogarithms and computed the monodromies for multiple logarithms. Firstly, let us describe the fundamental group of $S_d(\mathbb{C})$.

Theorem 2.3.22 ([15])**.** $\pi_1(S_d(\mathbb{C}))$ is the free group generated by loops around the divisors

$$\{x_r = 0\} \text{ and } \{x_j \cdots x_k = 1\}$$

More specifically, it is torsion free of rank $2d + \binom{d}{2}$, generated by ν_r, $1 \le r \le d$ and ν_{jk}, $1 \le j \le k \le d$, where ν_r is the positive loop around the divisor $x_r = 0$, i.e.

$$\int_{\nu_r} d\log(x_s) = 2\pi i\delta_{rs}, \quad \int_{\nu_i} d\log(1 - x_j \cdots x_k) = 0$$

and $\nu_{j,k}$ is the positive loop around the divisor $x_j \cdots x_k = 1$, i.e.

$$\int_{\nu_{j,k}} d\log(x_s) = 0, \quad \int_{\nu_{j,k}} d\log(1 - x_p \cdots x_q) = 2\pi i\delta_{jp}\delta_{kq}$$

Here δ is the Kronecker delta.

Zhao also gives concrete choices for $\nu_r, \nu_{j,k}$. For example, ν_r could be a loop in some plane $\{x_s = a_s \neq 0, \forall s \neq r\}$ such that

$$\oint_{\nu_r} \frac{dz}{z} = 2\pi i, \quad \oint_{\nu_{j,k}} \frac{dz}{z-1} = 0$$

and $\nu_{j,k}$ could be a loop in some plane $\{x_s = a_s \neq 0, \forall s \neq r\}$ where any products of a_s does not equal to 1, and $\nu_{j,k}$ satisfies

$$\oint_{\nu_r} \frac{dz}{z} = 0, \quad \oint_{\nu_{j,k}} \frac{dz}{z-1} = 2\pi i$$

Zhao's strategy is to convert these iterated integrals on $\mathbb{C}$ into iterated integrals on $S_d(\mathbb{C})$, and then use this to compute the monodromies of multiple logarithms.

Theorem 2.3.23. Multiple polylogarithms can be expressed as iterated integrals of logarithmic differential forms over $S_d(\mathbb{C})$.

Example 2.3.24.

$$\mathrm{Li}_{1,1}(x_1, x_2) = \int_{(0,0)}^{(1,1)} \frac{dx_2}{1-x_2}\frac{dx_1}{1-x_1} + \frac{d(x_1x_2)}{1-x_1x_2}\left(\frac{dx_2}{1-x_2} + \frac{dx_1}{x_1(x_1-1)}\right) \tag{2.3.16}$$

Let's denote the monodromy operator as $\mathcal{M}_{\nu_i}$ and $\mathcal{M}_{\nu_{j,k}}$. Zhao has the following theorem describing the monodromy of multiple logarithms - multiple polylogarithms $\mathrm{Li}_{n_1,\cdots,n_d}$ with $\mathbf{n} = (n_1, \cdots, n_d) = (1, \cdots, 1)$. In Chapter 5, we will generalize the theorem to arbitrary $\mathbf{n}$.

Theorem 2.3.25.

$$(\mathcal{M}_{\nu_i} - \mathrm{id})\,\mathrm{Li}_{1,\cdots,1}(x_1, \cdots, x_d) = 0, \quad 1 \leq i \leq d \tag{2.3.17}$$

$$(\mathcal{M}_{\nu_{j,k}} - \mathrm{id})\,\mathrm{Li}_{1,\cdots,1}(x_1, \cdots, x_d) = 0, \quad \forall 1 \leq j < k < d \tag{2.3.18}$$

$$(\mathcal{M}_{\nu_{j,j}} - \mathrm{id})\,\mathrm{Li}_{1,\cdots,1}(x_1,\cdots,x_d) = 0, \quad \forall 1 \le j < d \tag{2.3.19}$$

$$(\mathcal{M}_{\nu_{d,d}} - \mathrm{id})\,\mathrm{Li}_{1,\cdots,1}(x_1,\cdots,x_d) = 2\pi i\,\mathrm{Li}_{1,\cdots,1}(x_1,\cdots,x_{d-1}) \tag{2.3.20}$$

$$\begin{aligned}(\mathcal{M}_{\nu_{j,d}} - \mathrm{id})\,\mathrm{Li}_{1,\cdots,1}(x_1,\cdots,x_d) &= 2\pi i\,\mathrm{Li}_{1,\cdots,1}(x_1,\cdots,x_{j-1}) \\ &\mathrm{Li}_{1,\cdots,1}\left(\frac{1-x_jx_{j+1}}{1-x_j},\cdots,\frac{1-x_j\cdots x_d}{1-x_j\cdots x_{d-1}}\right), \quad \forall 1 \le j < d\end{aligned} \tag{2.3.21}$$

2.4 Variation of mixed Hodge structures

Multiple polylogarithms naturally define variations of mixed Hodge structures. These mixed Hodge structures are described by filtrations on the flat sections of a vector bundle, so first we need to review connections on vector bundles and how they are related to local systems with monodromies.

2.4.1 Connection on vector bundles

Definition 2.4.1. Suppose B is a complex manifold and $E \to B$ is a vector bundle. The E valued k-forms are defined to be

$$\Omega^k(E) = \Gamma\left(\bigwedge^k T^*B \otimes E\right) = \Omega^k(B) \otimes \Gamma(E)$$

a *vector bundle(Koszul) connection* is

$$\nabla : \Omega^0(E) \to \Omega^1(E) = T^*B \otimes \Gamma(E)$$

satisfying the Leibniz rule

$$\nabla(f \otimes s) = df \otimes s + f \wedge \nabla s \tag{2.4.1}$$

A vector field $X \in TB$ defines the covariant derivative $\nabla_X : \Gamma(E) \to \Gamma(E)$ along X, which by (2.4.1) uniquely extends to a exterior covariant derivative $\nabla : \Omega^k(E) \to \Omega^{k+1}(E)$ through

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^{|\omega|}\omega \wedge \nabla s$$

Definition 2.4.2 (Connection form)**.**

Over a local trivialization, suppose $s = \sum_i s_i \otimes e_i$, $\nabla e_i = \sum_j \omega_{ji} \otimes e_j$, then

$$\nabla s = \sum_i ds_i \otimes e_i + \sum_i s_i \wedge \left(\sum_j \omega_{ji} \otimes e_j\right) = \sum_i \left(ds_i + \sum_j \omega_{ij} \wedge s_j\right) \otimes e_i$$

In short, we have $\nabla s = ds + \omega \wedge s$, where $\omega = (\omega_{ij})$ is known as the *connection form*.

Definition 2.4.3 (Curvature)**.** The curvature tensor is defined as

$$\nabla^2 : \Omega^0(E) \to \Omega^2(E)$$

Over a local trivialization, $\nabla^2 s = \nabla(ds + \omega \wedge s) = (d\omega + \omega \wedge \omega) \wedge s$, thus $d\omega + \omega \wedge \omega$ is called the *curvature form*. A connection is *flat(completely integrable)* if its curvature is zero.

Proposition 2.4.4. If ϕ is an automorphism of $E \to B$, the conjugate $\nabla^\phi = \phi^{-1} \circ \nabla \circ \phi$ is again a connection. Furthermore, ∇^ϕ is flat if ∇ is flat.

Proof.

$$\nabla^\phi(f \otimes s) = \phi^{-1} \circ \nabla(f \otimes \phi s) = \phi^{-1}(df \otimes \phi s + f \wedge \nabla(\phi s)) = df \otimes s + f \wedge \nabla^\phi s$$

Over a local trivialization, $\phi s = As$. We then have

$$A^{-1}(\nabla(As)) = A^{-1}(d(As) + \omega \wedge As) = A^{-1}(dAs + Ads + \omega \wedge As)$$

$$= ds + (A^{-1}dA + A^{-1}\omega A)s$$

Hence the connection form for ∇^ϕ is $A^{-1}dA + A^{-1}\omega A$. If $d\omega + \omega \wedge \omega = 0$, then

$$d(A^{-1}dA + A^{-1}\omega A) + (A^{-1}dA + A^{-1}\omega A) \wedge (A^{-1}dA + A^{-1}\omega A) = 0$$

Hence, ∇^ϕ is a flat connection. $\square$

We can rephrase vector bundles as locally free sheaves.

Definition 2.4.5. Suppose B is a complex manifold and $E \to B$ is a vector bundle. The sheaf of sections of E is a locally free sheaf $\mathcal{E}$, and a connection ∇ on E can be naturally generalized as a connection on $\mathcal{E}$, via $\nabla : \Omega_B^k \otimes_{\mathcal{O}_B} \mathcal{E} \to \Omega_B^{k+1} \otimes_{\mathcal{O}_B} \mathcal{E}$

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^{|\omega|}\omega \wedge \nabla s$$

Note that locally $\Gamma(\Omega_U^k \otimes_{\mathcal{O}_U} \mathcal{E}_U) \cong \Omega^k(E|_U)$. So we can define connection form $\omega = (\omega_{ij})$ as well.

Theorem 2.4.6 (Riemann-Hilbert Correspondence)**.** Suppose B is a complex manifold. The category of locally constant sheaves of vector spaces is equivalent to the category of locally free sheaves with flat connections.

Proof. Suppose $\mathbb{V}$ is a locally constant sheaf, then $\mathcal{V} = \mathcal{O}_B \otimes_{\underline{\mathbb{Z}}} \mathbb{V}$ is a locally free sheaf, and $\nabla(f \otimes v) = df \otimes v$ defines a flat connection. On the other hand, suppose $\mathcal{E}$ is a locally free sheaf with a flat connection ∇ with connection form ω. Then the sheaf of flat sections forms a locally constant sheaf. If we take the boundary ∂R of a simple region R, and suppose f is a flat section so that $df + \omega \wedge f = 0$ locally, then

$$\int_{\partial R} df = \int_R d^2 f$$

But locally we have

$$\begin{aligned} d^2 f &= d(-\omega \wedge f) \\ &= \omega \wedge df - d\omega \wedge f \\ &= -\omega \wedge \omega \wedge f + \omega \wedge \omega \wedge f \\ &= 0 \end{aligned}$$

Therefore f can be well defined on the universal cover of B, thus giving a locally constant sheaf. □

Deligne [25] extended this result, a generalization necessary for constructing the variation of mixed Hodge structures in Chapter 4.

Theorem 2.4.7. Suppose $\overline{X}$ is a projective variety with $X \subseteq \overline{X}$ and $D = \overline{X} - X$ is a simple normal crossing. The category of locally constant sheaves of vector spaces is equivalent to locally free sheaves with connections.

2.4.2 Hodge structures and variations of mixed Hodge structures

Suppose H is a finitely generated abelian group, and denote $H_R = H \otimes R$ for any $\mathbb{Z}$-module R.

Definition 2.4.8. We say H has a pure Hodge structure of weight n if there is a finite decreasing Hodge filtration $F^\bullet$ of $H_\mathbb{C}$ by subspaces $\{F^p H_\mathbb{C}\}_{p\in\mathbb{Z}}$ such that

$$H_\mathbb{C} = F^p H_\mathbb{C} \oplus \overline{F^{n+1-p} H_\mathbb{C}}$$

If we denote $H^{p,n-p} = F^p H_{\mathbb{C}} \cap \overline{F^{n-p} H_{\mathbb{C}}}$, we then have the Hodge decomposition

$$H_{\mathbb{C}} = \bigoplus_{p+q=n} H^{p,q}$$

Definition 2.4.9. Suppose H is a finitely generated abelian group. We say it has a mixed Hodge structure if there is a finite decreasing Hodge filtration $F^\bullet$ of $H_{\mathbb{C}}$ by subspaces $\{F^p H_{\mathbb{C}}\}_{p\in\mathbb{Z}}$ and an increasing weight filtration $W^\bullet$ of $H_{\mathbb{Q}}$ by subspaces $\{W_k H_{\mathbb{Q}}\}_{k\in\mathbb{Z}}$ such that the graded piece $gr_k^W H_{\mathbb{Q}}$ is a pure Hodge structure of weight k with Hodge filtration given by

$$F^p(gr_k^W H_{\mathbb{Q}})_{\mathbb{C}} = \frac{F^p H_{\mathbb{C}} \cap (W_{k+1} H_{\mathbb{Q}})_{\mathbb{C}} + (W_k H_{\mathbb{Q}})_{\mathbb{C}}}{(W_k H_{\mathbb{Q}})_{\mathbb{C}}}$$

Let B be a complex manifold and $\mathbb{V}$ a locally constant sheaf of finitely generated abelian groups, then $\mathbb{V}_{\mathbb{Q}} = \mathbb{V} \otimes_{\underline{\mathbb{Z}}} \underline{\mathbb{Q}}$, $\mathbb{V}_{\mathbb{C}} = \mathbb{V} \otimes_{\underline{\mathbb{Z}}} \underline{\mathbb{C}}$ are locally constant sheaves of vector spaces, and $\mathcal{V} = \mathbb{V} \otimes_{\underline{\mathbb{Z}}} \mathcal{O}_B$ is a locally free sheaf.

Definition 2.4.10. A *variation of pure Hodge structures* of weight n is a decreasing filtration $F^\bullet$ of $\mathcal{V}$ by subsheaves $\{F^p\mathcal{V}\}$ such that each fiber $\mathbb{V}_b$ is a pure Hodge structure of weight n with Hodge filtration $\{F^p\mathbb{V}_{\mathbb{C},b}\}$ which satisfies Griffith transversality: $\nabla F^p\mathcal{V} \subseteq F^{p-1}\mathcal{V} \otimes \Omega_B^1$.

Definition 2.4.11. A *variation of mixed Hodge structures* consists of a decreasing filtration $F^\bullet$ of $\mathbb{V}$ by holomorphic subsheaves $\{F^p\mathcal{V}\}$, and a weight filtration $W_\bullet$ of $\mathbb{V}_{\mathbb{Q}}$ by subsheaves $\{W_k\mathbb{V}_{\mathbb{Q}}\}$ such that each fiber $\mathbb{V}_b$ is a mixed Hodge structure with Hodge filtration $F^p\mathbb{V}_{\mathbb{C},b}$ and weight filtration $W_k\mathbb{V}_{\mathbb{Q},b}$, again satisfying the Griffith transversality $\nabla F^p\mathcal{V} \subseteq F^{p-1}\mathcal{V} \otimes \Omega_B^1$.

2.5 Relation between iterated integrals and motivic cohomology

The following discussion can be found in [8].

Suppose $\mathcal{M}(F)$ is the category of mixed Tate motives over a number field F with pure Tate motives $\mathbb{Q}(n)$. The i-th motivic cohomology group $H^i_{\mathcal{M}}(F, \mathbb{Q}(n))$ is defined as i-th Ext group $\mathrm{Ext}^i_{\mathcal{M}(F)}(\mathbb{Q}(0), \mathbb{Q}(n))$. Since $\mathcal{M}(F)$ is a Tannakian category, $\mathcal{M}(F)$ is equivalent to the category of finite-dimension modules over $\mathrm{Aut}^{\otimes} \Psi$, where

$$\Psi : \mathcal{M}(F) \to \mathrm{Vect}_{\mathbb{Q}}, \quad M \mapsto \bigoplus_n \mathrm{Hom}(\mathbb{Q}(n), \mathrm{Gr}^W_{2n} M)$$

is the fiber functor, and $\mathrm{Aut}^{\otimes} \Phi$ is the group scheme of automorphisms of Φ respecting the tensor product.

$\mathrm{Aut}^{\otimes} \Psi$ is the semidirect product of the multiplicative group scheme $\mathbb{G}_m$ and a pro-unipotent group scheme $U(F)$, denote $L(F)$ as the Lie algebra of $U(F)$, $L(F)$ is graded due to the action of $\mathbb{G}_m$. Then $\mathcal{U}(F) = \mathrm{End}(\Psi)$ is isomorphic to the universal enveloping algebra of $L(F)$. Suppose $\mathcal{H}(F)$ is the dual of $\mathcal{U}(F)$, then $\mathcal{H}(F)$ can be identified with the graded Hopf algebra of regular functions on $U(F)$, and $\mathcal{L}(F) = \dfrac{\mathcal{H}_{>0}(F)}{\mathcal{H}_{>0}(F) \cdot \mathcal{H}_{>0}(F)}$ is a graded Lie coalgebra, the cotangent space of $U(F)$ which is dual of $L(F) = \mathrm{Lie}(U(F))$.

Goncharov constructed a Hopf algebra $\mathcal{A}(F)$ (see [8]) as the set of equivalences classes of framed mixed Tate motives $I^{\mathcal{M}}(a_0; a_1, \cdots, a_n; a_{n+1})$, with the coproduct similar to the coproduct (2.3.13) for iterated integrals, and showed that $\mathcal{A}(F)$ and $\mathcal{H}(F)$ are canonically isomorphic. He then considered its realizations in various categories, the benefit of this is that in each case we get a coproduct and Hopf algebra structure for free.

Chapter 3: Hopf Algebra of Multiple Polylogarithms and Its Associated One Forms

3.1 Two new Hopf algebras of multiple polylogarithms

In this section, we give the definitions of two new purely symbolic Hopf algebras $\overline{\mathbb{H}}^{\mathrm{Symb}}$ and $\mathbb{H}^{\mathrm{Symb}}$ with generators representing multiple polylogarithms. $\overline{\mathbb{H}}^{\mathrm{Symb}}$ is the smallest Hopf algebra that allows (3.1.2), a purely symbolic representation of the coproduct for multiple polylogarithms. $\mathbb{H}^{\mathrm{Symb}}$ is the smallest Hopf algebra that admits a differential and has no relations among its generators, this is explained in Lemma 3.1.13. This is worth noting that this differs from Goncharov and others various definitions of Hopf algebra of multiple polylogarithms, which typically assumes some polylogarithmic relations among its generators.

We will use the following shorthand throughout this chapter

$$x_{i\to j} := \prod_{i\leq k<j} x_k, \quad x^{-1}_{i\to j} := \prod_{i\leq k<j} x_k^{-1}, \quad 0^n := \overbrace{0,\cdots,0}^{k}$$

3.1.1 Definition of $\overline{\mathbb{H}}^{\mathrm{Symb}}$

Definition 3.1.1. $\overline{\mathbb{H}}^{\mathrm{Symb}}$ is the free $\mathbb{Q}$-algebra generated by two types of symbols:

$$\begin{aligned} \text{regular symbols:} \quad & [x_i]_0, [x_{i_1\to i_2}, x_{i_2\to i_3}, \cdots, x_{i_d\to i_{d+1}}]_{n_1,n_2,\cdots,n_d} \\ \text{inverted symbols:} \quad & [x^{-1}_{i_d\to i_{d+1}}, x^{-1}_{i_{d-1}\to i_d}, \cdots, x^{-1}_{i_1\to i_2}]_{n_d,n_{d-1},\cdots,n_1} \end{aligned}$$

where, $i, i_k, n_k \in \mathbb{Z}_{\geq 1}$ and $1 \leq i_1 < i_2 < \cdots < i_{d+1}$.

One should think of $[x_i]_0, [x_{i_1\to i_2}, \cdots, x_{i_d\to i_{d+1}}]_{n_1,\cdots,n_d}, [x^{-1}_{i_d\to i_{d+1}}, \cdots, x^{-1}_{i_1\to i_2}]_{n_d,\cdots,n_1}$ as $\log(x_i)$, $\mathrm{Li}_{n_1,\cdots,n_d}(x_{i_1\to i_2}, \cdots, x_{i_d\to i_{d+1}})$ and $\mathrm{Li}_{n_d,\cdots,n_1}(x^{-1}_{i_d\to i_{d+1}}, \cdots, x^{-1}_{i_1\to i_2})$ respectively.

If we associate a symbol with weight $n_1 + \cdots + n_d$ (1 for $[x_i]_0$), then $\overline{\mathbb{H}}^{\mathrm{Symb}}$ is graded with

- $\overline{\mathbb{H}}^{\mathrm{Symb}}_0 = \mathbb{Q}$
- $\overline{\mathbb{H}}^{\mathrm{Symb}}_1 = \mathbb{Q}\{[x_i]_0, [x_{i\to j}]_1\}$
- $\overline{\mathbb{H}}^{\mathrm{Symb}}_{\geq 2} = \mathbb{Q}\{\text{symbols with weights } \geq 2\}$

For convenience, we also define

$$[x_{i\to j}]_0 := \sum_{i\leq k<j} [x_k]_0, \quad [x^{-1}_{i\to j}]_0 := -\sum_{i\leq k<j} [x_k]_0$$

Inspired by Goncharov [8]. One could define a coproduct on $\overline{\mathbb{H}}^{\mathrm{Symb}}$. First denote the formal power series

$$[\mathbf{y}|\mathbf{t}] = [y_1, \ldots, y_d|t_1, \ldots, t_d] = \sum_{n_i\geq 1} [y_1, \ldots, y_d]_{n_1,\ldots,n_d} t_1^{n_1-1} \ldots t_d^{n_d-1} \tag{3.1.1}$$

Where $[y_1,\dots,y_d]_{n_1,\dots,n_d}$ are regular symbols with $y_k = x_{i_k\to i_{k+1}}$ for convenience. Then the coproduct formula is given by $\Delta([x_i]_0) = [x_i]_0\otimes 1 + 1\otimes [x_i]_0$, $\Delta([x^{\pm}_{i\to j}]_1) = [x^{\pm}_{i\to j}]_1\otimes 1 + 1\otimes [x^{\pm}_{i\to j}]_1$ and

$$\Delta([\mathbf{y}|\mathbf{t}]) = \sum [y_{i_1\to i_2},\dots,y_{i_k\to i_{k+1}}|t_{j_1},\dots,t_{j_k}] \bigotimes \prod_{\alpha=0}^{k} (-1)^{j_\alpha - i_\alpha} \exp([y_{i_\alpha\to i_{\alpha+1}}]_0 t_{j_\alpha})$$
$$[y^{-1}_{j_\alpha-1}, y^{-1}_{j_\alpha-2},\dots,y^{-1}_{i_\alpha}|t_{j_\alpha}-t_{j_\alpha-1},\dots,t_{j_\alpha}-t_{i_\alpha}]$$
$$[y_{j_\alpha+1}, y_{j_\alpha+2},\dots,y_{i_{\alpha+1}-1}|t_{j_\alpha+1}-t_{j_\alpha},\dots,t_{i_{\alpha+1}-1}-t_{j_\alpha}]. \tag{3.1.2}$$

The sum is over all instances of $1 = i_0 \le j_0 < i_1 \le j_1 < \cdots < i_k \le j_k < i_{k+1} = d+1$, and by definition we have $y_{i\to j} = \prod_{r=i}^{j-1} y_r$.

Theorem 3.1.2. $\overline{\mathbb{H}}^{\mathrm{Symb}}$ forms a graded Hopf algebra.

Proof. We justify the coassociativity of the coproduct in Corollary 3.1.19. The counit and antipode is defined by Lemma 2.2.3 and Theorem 2.2.6. $\square$

Example 3.1.3.

$$\Delta([x_1,x_2]_{1,1}) = 1\otimes [x_1,x_2]_{1,1} + [x_2]_1\otimes [x_1]_1 - [x_1x_2]_1\otimes [x_1^{-1}]_1$$
$$+ [x_1x_2]_1\otimes [x_2]_1 + [x_1,x_2]_{1,1}\otimes 1 \tag{3.1.3}$$

$$\Delta([x_1,x_2]_{2,1}) = 1\otimes [x_1,x_2]_{2,1} + [x_1,x_2]_{2,1}\otimes 1 - [x_1,x_2]_{1,1}\otimes [x_2]_0$$
$$+ [x_1,x_2]_{1,1}\otimes [x_1x_2]_0 + [x_2]_1\otimes [x_1]_2 + [x_1x_2]_1\otimes [x_1^{-1}]_2$$
$$- [x_1x_2]_1\otimes [x_2]_2 + [x_1x_2]_2\otimes [x_2]_1 + [x_1x_2]_1\otimes ([x_2]_1[x_1x_2]_0) \tag{3.1.4}$$

3.1.2 Definition of $\mathbb{H}^{\text{Symb}}$

The following is inspired by Goncharov's inversion (2.3.9) for multiple polylogarithms. It is natural to consider a map $\overline{\text{INV}}$ that converts inverted symbols into regular symbols. Let $\overline{\mathbb{H}}^{\text{Symb}}[\pi i]$ denote $\overline{\mathbb{H}}^{\text{Symb}}$ adjoined with a free generator πi.

Definition 3.1.4. $\overline{\text{INV}} : \overline{\mathbb{H}}^{\text{Symb}} \to \overline{\mathbb{H}}^{\text{Symb}}[\pi i]$ fixes regular symbols and acts on inverted symbols inductively by

$$\overline{\text{INV}}([x_d^{-1}, \cdots, x_1^{-1}]_{n_d, \cdots, n_1}) = -\sum_{r=0}^{d-1} \overline{\text{INV}}[x_r^{-1}, \cdots, x_1^{-1}]_{n_r, \cdots, n_1}[x_{r+1}, \cdots, x_d]_{n_{r+1}, \cdots, n_d} - \sum_{r=1}^{d} \sum_{m_1+\cdots+m_d=n_r} (-1)^{1+n_r+\cdots+n_d+m_{r+1}+\cdots+m_d} \prod_{\substack{1\leq i\leq d\\ i\neq r}} \binom{n_i+m_i-1}{n_i-1} B_{m_r}\left(\frac{[x_1\cdots x_d]_0}{2\pi i}\right) \frac{(2\pi i)^{m_r}}{m_r!} \overline{\text{INV}}[x_{r-1}^{-1}, \cdots, x_1^{-1}]_{n_{r-1}+m_{r-1}, \cdots, n_1+m_1}[x_{r+1}, \cdots, x_d]_{n_{r+1}+m_{r+1}, \cdots, n_d+m_d} \tag{3.1.5}$$

In depth 1 and 2, (2.3.10) and (2.3.10) correspond to

$$[x]_n + (-1)^n[x^{-1}]_n = -\frac{(2\pi i)^n}{n!} B_n\left(\frac{[x]_0}{2\pi i}\right), \tag{3.1.6}$$

$$\begin{aligned}
[x_1, x_2]_{n_1,n_2} + (-1)^{n_1+n_2}[x_2^{-1}, x_1^{-1}]_{n_2,n_1} + (-1)^{n_1}[x_1^{-1}]_{n_1}[x_2]_{n_2}\\
+ \sum_{p+q=n_1} \frac{(2\pi i)^p}{p!}(-1)^q \binom{q+n_2-1}{n_2-1} B_p\left(\frac{[x_1x_2]_0}{2\pi i}\right)[x_2]_{q+n_2}\\
+ \sum_{p+q=n_2} \frac{(2\pi i)^p}{p!}(-1)^{n_1} \binom{n_1+q-1}{n_1-1} B_p\left(\frac{[x_1x_2]_0}{2\pi i}\right)[x_1^{-1}]_{n_1+q} = 0,
\end{aligned} \tag{3.1.7}$$

It is difficult to work with powers of πi, so we also define INV as $\overline{\text{INV}}$ modulo πi.

Definition 3.1.5. The map $\text{INV} : \overline{\mathbb{H}}^{\text{Symb}} \to \overline{\mathbb{H}}^{\text{Symb}}$ fixes regular symbols and acts on inverted symbols inductively by

$$
\begin{aligned}
&\mathrm{INV}([y_d^{-1},\cdots,y_1^{-1}|-t_d,\cdots,-t_1])\\
&=\sum_{j=0}^{d-1}(-1)^{d-1+j}\,\mathrm{INV}([y_j^{-1},\cdots,y_1^{-1}|-t_j,\cdots,-t_1])[y_{j+1},\cdots,y_d|t_{j+1},\cdots,t_d]\\
&+\sum_{j=1}^{d}\frac{(-1)^{d-1+j}}{t_j}\,\mathrm{INV}([y_{j-1}^{-1},\cdots,y_1^{-1}|-t_{j-1},\cdots,-t_1])[y_{j+1},\cdots,y_d|t_{j+1},\cdots,t_d]\\
&+\sum_{j=1}^{d}\Bigg(\frac{(-1)^{d+j}}{t_j}\,\mathrm{INV}([y_{j-1}^{-1},\cdots,y_1^{-1}|t_j-t_{j-1},\cdots,t_j-t_1])\\
&\qquad\qquad \exp([y_{1\to d+1}]_0t_j)[y_{j+1},\cdots,y_d|t_{j+1}-t_j,\cdots,t_d-t_j]\Bigg)
\end{aligned}
\tag{3.1.8}
$$

with the induction starting with $\mathrm{INV}([y^{-1}|-t])=[y|t]+\dfrac{\exp([y]_0t)-1}{t}$.

Example 3.1.6. Corresponding to (3.1.6) and (3.1.7), we have

$$\mathrm{INV}([x_1^{-1}]_n)=(-1)^{n-1}[x_1]_n+\frac{(-1)^{n-1}}{n!}[x_1]^n,$$

$$
\begin{aligned}
\mathrm{INV}([x_2^{-1},x_1^{-1}]_{1,1})&=-[x_1,x_2]_{1,1}+[x_1]_1[x_2]_1-[x_1]_2+[x_2]_2+[x_2]_1[x_1]_0\\
&+[x_1]_1[x_1x_2]_0-[x_2]_1[x_1x_2]_0-\frac{1}{2}[x_1]_0^2+[x_1x_2]_0[x_1]_0
\end{aligned}
\tag{3.1.9}
$$

Definition 3.1.7. $\mathbb{H}^{\mathrm{Symb}}$ is the subalgebra of $\overline{\mathbb{H}}^{\mathrm{Symb}}$ generated solely by regular symbols $[x_i]_0$, $[x_{i_1\to i_2},\cdots,x_{i_d\to i_{d+1}}]_{n_1,\cdots,n_d}$.

Theorem 3.1.8. $\mathbb{H}^{\mathrm{Symb}}$ forms a graded Hopf algebra, with coproduct $\Delta_{\mathbb{H}}:\mathbb{H}^{\mathrm{Symb}}\to\mathbb{H}^{\mathrm{Symb}}\otimes\mathbb{H}^{\mathrm{Symb}}$ as the composition of the coproduct $\Delta_{\overline{\mathbb{H}}}$ of $\overline{\mathbb{H}}^{\mathrm{Symb}}$ and INV, to be precise, $\Delta_{\mathbb{H}}=(1\otimes\mathrm{INV})\circ\Delta_{\overline{\mathbb{H}}}$

Proof. This is precisely Theorem 3.1.20, combined with Corollary 3.1.19. ☐

Example 3.1.9.

$$\Delta([x_1,x_2]_{1,1}) = 1\otimes[x_1,x_2]_{1,1} + [x_2]_1\otimes[x_1]_1 - [x_1x_2]_1\otimes([x_1]_1+[x_1]_0) + [x_1x_2]_1\otimes[x_2]_1 + [x_1,x_2]_{1,1}\otimes 1 \quad (3.1.10)$$

$$\begin{aligned}\Delta([x_1,x_2]_{2,1}) &= 1\otimes[x_1,x_2]_{2,1} + [x_1,x_2]_{2,1}\otimes 1 - [x_1,x_2]_{1,1}\otimes[x_2]_0 \\ &+ [x_1,x_2]_{1,1}\otimes[x_1x_2]_0 + [x_2]_1\otimes[x_1]_2 + [x_1x_2]_1\otimes\left(-[x_1]_2 - \frac{1}{2}[x_1]_0^2\right) \\ &- [x_1x_2]_1\otimes[x_2]_2 + [x_1x_2]_2\otimes[x_2]_1 + [x_1x_2]_1\otimes([x_2]_1[x_1x_2]_0) \quad (3.1.11)\end{aligned}$$

3.1.3 Orderings on $\mathbb{H}^{\mathrm{Symb}}$

To order the terms in the coproduct formula 3.1.2 and in turn the variation matrix in chapter 4, we define a total ordering on the generators of $\mathbb{H}^{\mathrm{Symb}}$. Consider a total ordering on $\mathbb{Z}_{\geq 0}^{\infty} = \bigcup_\ell \mathbb{Z}_{\geq 0}^{\ell}$ where $\mathbf{k} \prec \mathbf{l}$ if

- $\|\mathbf{k}\| < \|\mathbf{l}\|$
- or if $\|\mathbf{k}\| = \|\mathbf{l}\|$ and $\dim(\mathbf{k}) < \dim(\mathbf{l})$
- or if $\|\mathbf{k}\| = \|\mathbf{l}\|$ and $\dim(\mathbf{k}) = \dim(\mathbf{l})$ and the rightmost nonzero entry of $\mathbf{l}-\mathbf{k}$ is negative.

Here $\|\mathbf{k}\| = \sum k_i$ is the weight for any $\mathbf{k} = (k_1,\cdots,k_\ell) \in \mathbb{Z}_{\geq 0}^{\infty}$.

Definition 3.1.10. We can impose a total ordering on the set of regular symbols by identifying it with $\mathbb{Z}_{\geq 0}^{\infty}$ via

$$[x_{i_1\to i_2},\cdots,x_{i_d\to i_{d+1}}]_{n_1,\cdots,n_d} \leftrightsquigarrow (0^{i_1-1}, n_1, 0^{i_2-i_1-1}, n_2, \cdots, 0^{i_d-i_{d-1}-1}, n_d, 0^{i_{d+1}-i_d-1})$$

Here 0^i means a tuple of i zeros.

Example 3.1.11.

$$0 \prec (0,1) \prec (1,0) \prec (0,2) \prec (1,1) \prec (2,0) \prec (0,3) \prec (1,2)$$

Corresponds to

$$1 \prec [x_2]_1 \prec [x_1x_2]_1 \prec [x_2]_2 \prec [x_1,x_2]_{1,1} \prec [x_1x_2]_1 \prec [x_2]_3 \prec [x_1,x_2]_{1,2}$$

3.1.4 Kähler differentials of $\mathbb{H}^{\mathrm{Symb}}$

Just like normal multiple polylogarithms, we can define differentials of elements of $\mathbb{H}^{\mathrm{Symb}}$.

Definition 3.1.12. We define linear maps $\partial_i : \mathbb{H}^{\mathrm{Symb}} \to \Omega_{\mathbb{H}^{\mathrm{Symb}}/\mathbb{Q}}$ as in Proposition 2.3.4. Thinking of $d[x_i]_0$ as $d\log(x_i) = \frac{dx_i}{x_i}$ and $d[x_i]_1$ as $d\,\mathrm{Li}_1(x_i) = \frac{dx_i}{1-x_i}$. We have

$$\partial_i[x_1,\cdots,x_d]_{n_1,\cdots,n_d} = [x_1,\cdots,x_d]_{n_1,\cdots,n_i-1,\cdots,n_d}d[x_i]_0, \quad n_i \geq 2 \tag{3.1.12}$$

$$\partial_d[x_1,\cdots,x_d]_{n_1,\cdots,1} = -[x_1,\cdots,x_{d-1}x_d]_{n_1,\cdots,n_{d-1}}d[x_d]_1 \tag{3.1.13}$$

$$\begin{aligned}\partial_1[x_1,\cdots,x_n]_{1,\cdots,n_d} = ([x_1x_2,\cdots,x_d]_{n_2,\cdots,n_d} - [x_2,\cdots,x_d]_{n_2,\cdots,n_d})d[x_1]_1 \\ - [x_1x_2,\cdots,x_d]_{n_2,\cdots,n_d}d[x_1]_0\end{aligned} \tag{3.1.14}$$

$$\begin{aligned}\partial_i[x_1,\cdots,x_d]_{n_1,\cdots,1,\cdots,n_d} &= -[x_1,\cdots,x_ix_{i+1},\cdots,x_d]_{n_1,\cdots,\widehat{1},\cdots,n_d}d[x_i]_0 \\ &+ ([x_1,\cdots,x_ix_{i+1},\cdots,x_d]_{n_1,\cdots,\widehat{1},\cdots,n_d} - [x_1,\cdots,x_{i-1}x_i,\cdots,x_d]_{n_1,\cdots,\widehat{1},\cdots,n_d})d[x_i]_1\end{aligned} \tag{3.1.15}$$

The differential $d : \mathbb{H}^{\mathrm{Symb}} \to \Omega_{\mathbb{H}^{\mathrm{Symb}}/\mathbb{Q}}$ is defined as the sum $\sum_i \partial_i$.

The following result is due to the author (not included in [2]). It is useful in that it shows that there are no formal relations among generators. It can be thought of as an analog of the classical theorem which says polynomials with zero differential must be constants.

Lemma 3.1.13. Consider the differential $d : \mathbb{H}^{\mathrm{Symb}}[\pi i] \to \Omega_{\mathbb{H}^{\mathrm{Symb}}[\pi i]/\mathbb{Q}[\pi i]}$, if $df = 0$, then $f \in \mathbb{Q}[\pi i]$ must be a constant.

Proof. The proof can be divided into several steps.

1. If we write $f = f_0 + (\pi i) f_1 + \cdots$, where $f_i \in \mathbb{H}^{\mathrm{Symb}}$, then $df = 0 \iff df_i = 0, \forall i$. So we may assume $f \in \mathbb{H}^{\mathrm{Symb}}$.

2. Let $\mathbb{H}^{\mathrm{Symb}}_{|\mathbf{n}|\geq 2} \subset \mathbb{H}^{\mathrm{Symb}}$ be the subalgebra generated by regular symbols of weight no less than 2, and denote for short $[\mathbf{x}]_0^{\mathbf{p}} = [x_1]_0^{p_1} \cdots [x_d]_0^{p_d}$, $[\mathbf{x}]_1^{\mathbf{q}} = \prod\limits_{1\leq j\leq k\leq d} [x_{j\to k}]_1^{q_{jk}}$. Write f as

$$f = \sum_{p_1,\cdots,p_d,q_1,\cdots,q_d\geq 0} f_{\mathbf{p},\mathbf{q}}[\mathbf{x}]_0^{\mathbf{p}}[\mathbf{x}]_1^{\mathbf{q}}, \qquad f_{\mathbf{p},\mathbf{q}} \in \mathbb{H}^{\mathrm{Symb}}_{|\mathbf{n}|\geq 2}$$

We will show that $f \in \mathbb{H}^{\mathrm{Symb}}_{|\mathbf{n}|\geq 2}$. Pick a term with maximal $|\mathbf{p}| + |\mathbf{q}| \neq 0$. We may assume either $p_i \neq 0$ for some i or $q_{jk} \neq 0$ for some $j < k$. If we denote $\mathbf{p}' = (p_1, \cdots, p_i - 1, \cdots, p_d)$, $\mathbf{q}' = (q_{12}, \cdots, q_{jk} - 1, \cdots, q_{d(d+1)})$, then we have either

$$0 = df = df_{\mathbf{p},\mathbf{q}}[\mathbf{x}]_0^{\mathbf{p}}[\mathbf{x}]_1^{\mathbf{q}} + (p_i f_{\mathbf{p},\mathbf{q}} d[x_i]_0 + df_{\mathbf{p}',\mathbf{q}})[\mathbf{x}]_0^{\mathbf{p}'}[\mathbf{x}]_1^{\mathbf{q}} + \cdots$$

or

$$0 = df = df_{\mathbf{p},\mathbf{q}}[\mathbf{x}]_0^{\mathbf{p}}[\mathbf{x}]_1^{\mathbf{q}} + (q_{jk} f_{\mathbf{p},\mathbf{q}} d[x_{j\to k}]_1 + df_{\mathbf{p},\mathbf{q}'})[\mathbf{x}]_0^{\mathbf{p}}[\mathbf{x}]_1^{\mathbf{q}'} + \cdots$$

This implies $df_{\mathbf{p},\mathbf{q}} = 0$, so $f_{\mathbf{p},\mathbf{q}}$ is a nonzero constant by induction on weight. But then $p_i f_{\mathbf{p},\mathbf{q}} d[x_i]_0 + df_{\mathbf{p}',\mathbf{q}}$ and $q_{jk} f_{\mathbf{p},\mathbf{q}} d[x_{j\to k}]_1 + df_{\mathbf{p},\mathbf{q}'}$ cannot be zero.

3. Recall the ordering $\prec$ in 3.1.3. If we denote the greatest variable of f by L, then we can write $f = L^m f_m + L^{m-1} f_{m-1} + \cdots + f_0$, $f_m \neq 0, m \geq 1$. Hence we get

$$0 = df = L^m df_m + L^{m-1}(mf_m dL + df_{m-1}) + \cdots + df_0$$

Again, by induction on the weight, $df_m = 0$ implies that f_m is a nonzero constant. Let us show that $mf_m dL + df_{m-1}$ cannot be zero. For this, we only need to show that dL is not equal to dg for any $g \in \mathbb{H}^{\mathrm{Symb}}_{|\mathbf{n}|\geq 2}$ with variables strictly less than L. Now suppose L corresponds to

$$(\cdots, \overset{\substack{i_{d-1}\text{-th}\\ \downarrow}}{n_{d-1}}, 0, \cdots, 0, \overset{\substack{i_d\text{-th}\\ \downarrow}}{n_d}, 0, \cdots, 0)$$

and L' corresponds to

$$(\cdots, n_{d-1}, 0, \cdots, 0, n_d - 1, 0, \cdots, 0)$$

Then dL contains term $L' d[x_{i_{d-1}\to i_{d+1}}]_1$ or $L' d[x_{i_{d-1}\to i_{d+1}}]_0$ which cannot be a term in dg.

□

Remark 3.1.14. Multiple polylogarithms satisfy stuffle relations [9] such as

$$\mathrm{Li}_{1,1}(x_1, x_2) + \mathrm{Li}_{1,1}(x_2, x_1) + \mathrm{Li}_2(x_1 x_2) - \mathrm{Li}_1(x_1)\,\mathrm{Li}_1(x_2) = 0 \tag{3.1.16}$$

so one should have

$$d([x_1, x_2]_{1,1} + [x_2, x_1]_{1,1} + [x_1 x_2]_2 - [x_1]_1 [x_2]_1) = 0$$

At first glance, this might seem contradictory, however, this won't be an issue since symbols like $[x_2, x_1]_{1,1}$ with reversed order in x's do not exist in $\overline{\mathbb{H}}$. This Lemma allows us to regard $\mathbb{H}^{\text{Symb}}$ as a minimally generated Hopf algebra with symbolic multiple polylogarithmic generators, without assuming any relations.

The differential of the generating series $[x_1, \ldots, x_n | t_1, \ldots, t_n]$ is straightforward to compute, and is given by:

$$
\begin{aligned}
d[x_1, \ldots, x_n | t_1, \ldots, t_n] = [x_1, \ldots, x_n | t_1, \ldots, t_n] & \left(\sum_{k=1}^{n} d[x_k]_0 t_k \right) \\
& + [x_2, \ldots, x_n | t_2, \ldots, t_n] d[x_1]_1 \\
& + \sum_{k=2}^{n} [x_1, \ldots, x_{k-1} x_k, \ldots, x_n | t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n) d[x_k]_1 \\
& - \sum_{k=1}^{n-1} [x_1, \ldots, x_k x_{k+1}, \ldots, x_n | t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n] \left(d[x_k]_1 + d[x_k]_0 \right)
\end{aligned}
\tag{3.1.17}
$$

3.1.5 Morphism from iterated integrals to $\mathbb{H}^{\text{Symb}}$

In this section, we give a proof of Theorem 3.1.2 and Theorem 3.1.8 using Goncharov's coproduct 3.1.2 on plane trees. Note that this is different than the proof in [8] because our proof doesn't assume shuffle relations among the generators of $\overline{\mathbb{H}}^{\text{Symb}}$. We achieve this by constructing a Hopf algebra morphism Φ from a certain Hopf subalgebra of $\mathcal{I}(S)$ to $\overline{\mathbb{H}}^{\text{Symb}}$.

Out of all iterated integrals, we are particularly interested in those that can be turned in to standard multiple polylogarithms. This is formalized as follows.

Definition 3.1.15. Let $S = \{0, 1\} \cup \{x_{i \to j}^{-1}\}_{i<j}$. Define $\mathbb{I}^{\text{Symb}}$ is defined to be the

Hopf subalgebra of $\mathcal{I}(S)$ generated by elements of the forms

$$I(0;0^{n_0-1},x^{-1}_{i_1\to m},0^{n_1-1},\cdots,x^{-1}_{i_d\to m},0^{n_d-1};x^{-1}_{i_{d+1}\to m}),$$

$$I(x^{-1}_{i_1\to m};0^{n_1-1},\cdots,x^{-1}_{i_d\to m},0^{n_d-1};0),\quad I(0;0^{n_d-1},x^{-1}_{i_d\to m},\cdots,0^{n_1-1};x^{-1}_{i_1\to m}),$$

$$I(x^{-1}_{i_1\to m},0^{n_1-1},\cdots,x^{-1}_{i_d\to m},0^{n_d-1};x^{-1}_{i_{d+1}\to m})$$

for any $1\le i_1<\cdots<i_d<i_{d+1}\le m$ and $n_k\in\mathbb{Z}_{\ge 1}$. Note $x^{-1}_{i_{d+1}\to m}$ could be 1 if $i_{d+1}=m$.

Finally, we are ready to construct the Hopf algebra morphism $\Phi:\mathbb{I}^{\text{Symb}}\to\overline{\mathbb{H}}^{\text{Symb}}$. Its definition is based on the properties of iterated integrals.

Definition 3.1.16. [2] Suppose $a_i\neq 0$ and denote $[1]_0=0$. Then $\Phi:\mathbb{I}^{\text{Symb}}\to\overline{\mathbb{H}}^{\text{Symb}}$ is defined on the generating series as follows

1.

$$\begin{aligned}\Phi(I(0;&a_1,\cdots,a_m;a_{m+1}|t_0,\cdots,t_m))=\\&(-1)^m\exp([a_{m+1}]_0t_0)\left[\frac{a_2}{a_1},\cdots,\frac{a_{m+1}}{a_m}\middle|t_1-t_0,\cdots,t_m-t_0\right]\end{aligned}\tag{3.1.18}$$

2.

$$\Phi(I(a_0;a_1,\cdots,a_m;0|t_0,\cdots,t_m))=(-1)^m\Phi(I(0;a_m,\cdots,a_1;a_0|-t_m,\cdots,-t_1))\tag{3.1.19}$$

3.

$$\begin{aligned}\Phi(I(a_0;\cdots;a_{m+1}|t_0,\cdots,t_m))=\prod_{p=0}^{k}\Phi(I(a_{i_p};a_{i_p+1},\cdots,a_{j_p};0|t_{i_p},\cdots,t_{j_p}))\\ \Phi(I(0;a_{j_p+1},\cdots,a_{i_{p+1}-1};a_{i_{p+1}}|t_{j_p},\cdots,t_{i_{p+1}-1}))\end{aligned}\tag{3.1.20}$$

Or equivalently, Φ is defined on the generators by

$$\Phi(I(0;0^{n_0-1},a_1,0^{n_1-1},\cdots,a_m,0^{n_m-1};a_{m+1}))=\sum_{i_0+\cdots+i_m=n_0-1}(-1)^{n_0+i_0+m-1}$$
$$\frac{[a_{m+1}]_0^{i_0}}{i_0!}\binom{n_1+i_1-1}{n_1-1}\cdots\binom{n_m+i_m-1}{n_m-1}\left[\frac{a_2}{a_1},\cdots,\frac{a_{m+1}}{a_m}\right]_{n_1+i_1,\cdots,n_m+i_m} \tag{3.1.21}$$

$$\Phi(I(a_0;0^{n_0-1},\cdots,a_m,0^{n_m-1};0))=(-1)^{n_0+\cdots+n_m-1}\Phi(I(0;0^{n_m-1},a_m,\cdots,0^{n_0-1};a_0)) \tag{3.1.22}$$

$$\Phi(I(a_0;0^{n_0-1},\cdots,a_m,0^{n_m-1};a_{m+1}))=$$
$$\sum_{k=0}^{m}\sum_{p+q=n_k>1}\Phi(I(a_0;\cdots,a_k,0^{p-1};0))\Phi(I(0,0^{q-1},a_{k+1},\cdots;a_{m+1})) \tag{3.1.23}$$

Example 3.1.17.

$$\Phi(I(0;0^n;x_1^{-1}))=\frac{(-1)^n}{n!}[x_1]_0$$

$$\Phi(I(0;x_{i_1\to i_{d+1}}^{-1},0^{n_1-1},\cdots,x_{i_d\to i_{d+1}}^{-1},0^{n_d-1};1))=(-1)^d[x_{i_1\to i_2},\cdots,x_{i_d\to i_{d+1}}]_{n_1,\cdots,n_d}$$

$$\Phi(I(0;0{,}0,(x_1x_2)^{-1},0;x_2^{-1}))=-\frac{1}{2}[x_2]_0^2[x_1]_2-[x_2]_0[x_1]_3-[x_1]_4$$

$$\Phi((x_1x_2)^{-1};0,0;x_2^{-1})=\frac{1}{2}[x_2]_0^2-[x_1x_2]_0[x_2]_0+\frac{1}{2}[x_1x_2]_0^2$$

Theorem 3.1.18. [2] Φ is a Hopf algebra homomorphism.

Proof. The proof can be found in [2], Proposition 6.12. □

Since Φ preserves coproduct, immediately we have

Corollary 3.1.19. The coproduct on $\overline{\mathbb{H}}^{\mathrm{Symb}}$ is coassociative.

To prove Theorem 3.1.8, we only need to show that coproduct commutes with INV. This fact is straightforward yet lengthy and tedious. We put its proof in the next section.

3.1.6 Commutativity of the coproduct and INV

Theorem 3.1.20. The map INV : $\overline{\mathbb{H}}^{\mathrm{Symb}} \to \mathbb{H}^{\mathrm{Symb}}$ is a homomorphism of Hopf algebras, i.e. $\Delta_{\mathbb{H}} \circ \mathrm{INV} = \mathrm{INV} \circ \Delta_{\overline{\mathbb{H}}}$.

Proof. $\Delta \circ \mathrm{INV} = \mathrm{INV} \circ \Delta$ clearly holds for the regular terms, so we consider only inverse terms. Assume this holds for lower depth (the depth one case $\mathrm{INV} \circ \Delta[y^{-1}|-t] = \Delta \circ \mathrm{INV}[y^{-1}|-t]$ is elementary). By the definition of INV (3.1.8) one has $\mathrm{INV}[y_d^{-1}, \dots, y_1^{-1}|-t_d, \dots, -t_1] = \sum \mathrm{INV}\, A_i$ with all A_i of lower depth. By induction, we thus have

$$\Delta \circ \mathrm{INV}[y_d^{-1} \cdots, y_1^{-1}| - t_d, \cdots, -t_1] = \sum \Delta \circ \mathrm{INV}\, A_i = \sum \mathrm{INV} \circ \Delta A_i, \quad (3.1.24)$$

so it suffices to show that $\sum \mathrm{INV} \circ \Delta A_i = \mathrm{INV} \circ \Delta [y_d^{-1}, \dots, y_1^{-1}| - t_d, \dots, -t_1]$. Rearranging the terms this is equivalent to:

$$\begin{aligned} 0 = \mathrm{INV} \circ \Delta \Bigg(& \sum_{j=0}^{d} (-1)^j [y_j^{-1}, \cdots, y_1^{-1}| - t_j, \cdots, -t_1][y_{j+1}, \cdots, y_d|t_{j+1}, \cdots, t_d] \\ & + \sum_{j=1}^{d} \frac{(-1)^j}{t_j} [y_{j-1}^{-1}, \cdots, y_1^{-1}| - t_{j-1}, \cdots, -t_1][y_{j+1}, \cdots, y_d|t_{j+1}, \cdots, t_d] \\ & - \sum_{j=1}^{d} \frac{(-1)^j}{t_j} [y_{j-1}^{-1}, \cdots, y_1^{-1}|t_j - t_{j-1}, \cdots, t_j - t_1] \\ & \exp([y_{1 \to d+1}]_0 t_j)[y_{j+1}, \cdots, y_d|t_{j+1} - t_j, \cdots, t_d - t_j] \Bigg). \end{aligned} \quad (3.1.25)$$

We must prove (3.1.25). We write the right hand side of (3.1.25) as $\mathrm{INV} \circ \Delta(A + B - C)$ and rewrite $\exp([X]_0 t)$ as X^t. In what follows we use some natural notational simplifications, e.g. we write $y_{j,\dots,i}^{-1}$ instead of $(y_j^{-1}, y_{j-1}^{-1}, \dots, y_i^{-1})$ and $t_k - t_{j,\dots,i}$ instead of $(t_k - t_j, t_k - t_{j-1}, \dots, t_k - t_i)$. We first rewrite $\Delta(B)$ as

$$\sum_{r=1}^{d} \frac{(-1)^r}{t_r} \Delta[y^{-1}_{r-1,\dots,1}| - t_{r-1,\dots,1}] \Delta[y_{r+1,\dots,d}|t_{r+1,\dots,d}]$$

$$= \sum_{r=1}^{d} \frac{(-1)^r}{t_r} \sum_{1=i_0 \le j_0 < \cdots < i_q \le r < i_{q+1} < \cdots < i_{k+1} = d+1} [y^{-1}_{i_{q-1}\to i_q,\dots,i_0\to i_1}| - t_{j_{q-1},\dots,j_0}]$$

$$[y_{i_{q+1}\to i_{q+2},\dots,i_k \to i_{k+1}}|t_{j_{q+1},\dots,k}] \otimes [y^{-1}_{r-1,\dots,i_q}| - t_{r-1,\dots,i_q}][y_{r+1,\dots,i_{q+1}-1}|t_{r+1,\dots,i_{q+1}-1}]$$

$$\prod_{p=0}^{q-1} (-1)^{j_p - i_{p+1}+1} y^{t_{j_p}}_{i_p \to i_{p+1}} [y_{j_p+1,\dots,i_{p+1}-1}|t_{j_p+1,\dots,i_{p+1}-1} - t_{j_p}][y^{-1}_{j_p-1,\dots,i_p}|t_{j_p} - t_{j_p-1,\dots,i_p}]$$

$$\prod_{p=q+1}^{k} (-1)^{j_p - i_p} y^{t_{j_p}}_{i_p \to i_{p+1}} [y^{-1}_{j_p-1,\dots,i_p}|t_{j_p} - t_{j_p-1,\dots,i_p}][y_{j_p+1,\dots,i_{p+1}-1}|t_{j_p+1,\dots,i_{p+1}-1} - t_{j_p}],$$

which simplifies to

$$\sum_{1=i_0 \le j_0 < \cdots < i_q \le j_q < i_{q+1} < \cdots < i_{k+1} = d+1} \frac{(-1)^{j_q}}{t_{j_q}} [y^{-1}_{i_{q-1}\to i_q,\dots,i_0 \to i_1}| - t_{j_{q-1},\dots,j_0}]$$

$$[y_{i_{q+1}\to i_{q+2},\dots,i_k\to i_{k+1}}|t_{j_{q+1},\dots,j_k}] \otimes [y^{-1}_{j_q-1,\dots,i_q}| - t_{j_q-1,\dots,i_q}][y_{j_q+1,\dots,i_{q+1}-1}|t_{j_q+1,\dots,i_{q+1}-1}](-1)^{i_q+q+1}$$

$$\prod_{0 \le p \le k, p \ne q} (-1)^{j_p - i_p} y^{t_{j_p}}_{i_p \to i_{p+1}} [y^{-1}_{j_p-1,\dots,i_p}|t_{j_p} - t_{j_p-1,\dots,i_p}][y_{j_p+1,\dots,i_{p+1}-1}|t_{j_p+1,\dots,i_{p+1}-1} - t_{j_p}]. \tag{3.1.26}$$

Similarly, $\Delta(C)$ can be simplified to

$$\sum_{1 \le i_0 \le j_0 < \cdots < i_q \le j_q < i_{q+1} < \cdots < i_{k+1} = d+1} \frac{(-1)^{j_q}}{t_{j_q}} [y^{-1}_{i_{q-1}\to i_q,\dots,i_0\to i_1}|t_{j_q} - t_{j_{q-1},\dots,j_0}] y^{t_{j_q}}_{1\to d+1}$$

$$[y_{i_{q+1}\to i_{q+2},\dots,i_k \to i_{k+1}}|t_{j_{q+1},\dots,j_k} - t_{j_q}] \otimes [y^{-1}_{j_q-1,\dots,i_q}, t_{j_q} - t_{j_q-1,\dots,i_q}] y^{t_{j_q}}_{1\to d+1}$$

$$[y_{j_q+1,\dots,i_{q+1}-1}|t_{j_q+1,\dots,i_{q+1}-1} - t_{j_q}](-1)^{i_q+q+1}$$

$$\prod_{0 \le p \le k, p \ne q} (-1)^{j_p - i_p} y^{t_{j_p} - t_{j_q}}_{i_p \to i_{p+1}} [y^{-1}_{j_p-1,\dots,i_p}|t_{j_p} - t_{j_p-1,\dots,i_p}][y_{j_p+1,\dots.i_{p+1}-1}|t_{j_p+1,\dots.i_{p+1}-1} - t_{j_p}] \tag{3.1.27}$$

Finally, $\Delta(A)$ equals

$$\begin{aligned}
&\sum_{r=0}^{d}(-1)^r\Delta[y^{-1}_{r,\dots,1}|-t_{r,\dots,1}]\Delta[y_{r+1,\dots,d}|t_{r+1,\dots,d}]\\
&\quad=\sum_{r=0}^{d}(-1)^r\sum_{1=i_0\le j_0<\dots<i_q\le r+1\le i_{q+1}<\dots<i_{k+1}=d+1}[y^{-1}_{i_{q-1}\to i_q,\dots,i_0\to i_1}|-t_{j_{q-1},\dots,j_0}]\\
&\qquad[y_{i_{q+1}\to i_{q+2},\dots,i_k\to i_{k+1}}|t_{j_{q+1},\dots,j_k}]\otimes[y^{-1}_{r,\dots,i_q}|-t_{r,\dots,i_q}][y_{r+1,\dots,i_{q+1}-1}|t_{r+1,\dots,i_{q+1}-1}]\\
&\qquad\prod_{p=0}^{q-1}(-1)^{j_p-i_{p+1}+1}y^{t_{j_p}}_{i_p\to i_{p+1}}[y_{j_p+1,\dots,i_{p+1}-1}|t_{j_p+1,\dots,i_{p+1}-1}-t_{j_p}][y^{-1}_{j_p-1,\dots,i_p}|t_{j_p}-t_{j_p-1,\dots,i_p}]\\
&\qquad\prod_{p=q+1}^{k}(-1)^{j_p-i_p}y^{t_{j_p}}_{i_p\to i_{p+1}}[y^{-1}_{j_p-1,\dots,i_p}|t_{j_p}-t_{j_p-1,\dots,i_p}][y_{j_p+1,\dots,i_{p+1}-1}|t_{j_p+1,\dots,i_{p+1}-1}-t_{j_p}],
\end{aligned}\tag{3.1.28}$$

which simplifies to

$$\begin{aligned}
&\sum_{1=i_0\le j_0<\dots<i_q\le i_{q+1}<\dots<i_{k+1}=d+1}[y^{-1}_{i_{q-1}\to i_q,\dots,i_0\to i_1}|-t_{j_{q-1},\dots,j_0}]\\
&\qquad[y_{i_{q+1}\to i_{q+2},\dots,i_k\to i_{k+1}}|t_{j_{q+1},\dots,j_k}]\bigotimes(-1)^q\\
&\qquad\left(\sum_{i_q\le r+1\le i_{q+1}}(-1)^{r-i_q+1}[y^{-1}_{r,\dots,i_q}|-t_{r,\dots,i_q}][y_{r+1,\dots,i_{q+1}-1}|t_{r+1,\dots,i_{q+1}-1}]\right)\\
&\qquad\prod_{0\le p\le k,p\neq q}(-1)^{j_p-i_p}y^{t_{j_p}}_{i_p\to i_{p+1}}[y^{-1}_{j_p-1,\dots,i_p}|t_{j_p}-t_{j_p-1,\dots,i_p}][y_{j_p+1,\dots,i_{p+1}-1}|t_{j_p+1,\dots,i_{p+1}-1}-t_{j_p}].
\end{aligned}\tag{3.1.29}$$

We split the sum into two parts depending on whether or not $i_q=i_{q+1}$:

$$\sum_{1=i_0\le j_0<\dots<i_q<i_{q+1}<\dots<i_{k+1}=d+1},\qquad\sum_{1=i_0\le j_0<\dots<i_q=i_{q+1}<\dots<i_{k+1}=d+1}\tag{3.1.30}$$

We then apply INV and use induction on the bracket of (3.1.29). The first sum becomes

$$\mathrm{INV}\Bigg\{\sum_{1=i_0\le j_0<\cdots<i_q<i_{q+1}<\cdots<i_{k+1}=d+1}[y^{-1}_{i_{q-1}\to i_q,\ldots,i_0\to i_1}|-t_{j_{q-1},\ldots,j_0}]$$

$$[y_{i_{q+1}\to i_{q+2},\ldots,i_k\to i_{k+1}}|t_{j_{q+1},\ldots,j_k}]\bigotimes(-1)^q$$

$$\Bigg(-\sum_{i_q\le r\le i_{q+1}-1}\frac{(-1)^{r-i_q+1}}{t_r}[y^{-1}_{r-1,\ldots,i_q}|-t_{r-1,\ldots,i_q}][y_{r+1,\ldots,i_{q+1}-1}|t_{r+1,\ldots,i_{q+1}-1}]$$

$$+\sum_{i_q\le r\le i_{q+1}-1}\frac{(-1)^{r-i_q+1}}{t_r}[y^{-1}_{r-1,\ldots,i_q}|t_r-t_{r-1,\ldots,i_q}]y^{t_r}_{i_q\to i_{q+1}}[y_{r+1,\ldots,i_{q+1}-1}|t_{r+1,\ldots,i_{q+1}-1}-t_r]\Bigg)$$

$$\prod_{0\le p\le k,p\ne q}(-1)^{j_p-i_p}y^{t_{j_p}}_{i_p\to i_{p+1}}[y^{-1}_{j_p-1,\ldots,i_p}|t_{j_p}-t_{j_p-1,\ldots,i_p}][y_{j_p+1,\ldots,i_{p+1}-1}|t_{j_p+1,\ldots,i_{p+1}-1}-t_{j_p}]\Bigg\}. \tag{3.1.31}$$

This equals

$$=-\mathrm{INV}\Bigg\{\sum_{1=i_0\le j_0<\cdots<i_q\le j_q<i_{q+1}<\cdots<i_{k+1}=d+1}\frac{(-1)^{j_q-i_q+q+1}}{t_{j_q}}[y^{-1}_{i_{q-1}\to i_q,\ldots,i_0\to i_1}|-t_{j_{q-1},\ldots,j_0}]$$

$$[y_{i_{q+1}\to i_{q+2},\ldots,i_k\to i_{k+1}}|t_{j_{q+1},\ldots,j_k}]\otimes[y^{-1}_{j_q-1,\ldots,i_q}|-t_{j_q-1,\ldots,i_q}][y_{j_q+1,\ldots,i_{q+1}-1}|t_{j_q+1,\ldots,i_{q+1}-1}]$$

$$\prod_{0\le p\le k,p\ne q}(-1)^{j_p-i_p}y^{t_{j_p}}_{i_p\to i_{p+1}}[y^{-1}_{j_p-1,\ldots,i_p}|t_{j_p}-t_{j_p-1,\ldots,i_p}][y_{j_p+1,\ldots,i_{p+1}-1}|t_{j_p+1,\ldots,i_{p+1}-1}-t_{j_p}]\Bigg\}$$

$$+\mathrm{INV}\Bigg\{\sum_{1=i_0\le j_0<\cdots<i_q\le j_q<i_{q+1}<\cdots<i_{k+1}=d+1}\frac{(-1)^{j_q-i_q+q+1}}{t_{j_q}}[y^{-1}_{i_{q-1}\to i_q,\ldots,i_0\to i_1}|-t_{j_{q-1},\ldots,j_0}]$$

$$[y_{i_{q+1}\to i_{q+2},\ldots,i_k\to i_{k+1}}|t_{j_{q+1},\ldots,j_k}]\otimes[y^{-1}_{j_q-1,\ldots,i_q}|t_{j_q}-t_{j_q-1,\ldots,i_q}][y_{j_q+1,\ldots,i_{q+1}-1}|t_{j_q+1,\ldots,i_{q+1}-1}-t_{j_q}]$$

$$y^{t_{j_q}}_{i_q\to i_{q+1}}\prod_{\substack{0\le p\le k\\ p\ne q}}(-1)^{j_p-i_p}y^{t_{j_p}}_{i_p\to i_{p+1}}[y^{-1}_{j_p-1,\ldots,i_p}|t_{j_p}-t_{j_p-1,\ldots,i_p}][y_{j_p+1,\ldots,i_{p+1}-1}|t_{j_p+1,\ldots,i_{p+1}-1}-t_{j_p}]\Bigg\}, \tag{3.1.32}$$

which we write as $-\operatorname{INV}(T_1)+\operatorname{INV}(T_2)$. The second sum becomes

$$\operatorname{INV}\left\{\sum_{1=i_1\le j_1<\cdots<i_{k+1}=d+1}\left(\right.\right.$$

$$\left.\sum_{0\le q\le k}(-1)^q[y^{-1}_{i_q\to i_{q+1},\dots,i_1\to i_2}|-t_{j_q,\dots,j_1}][y_{i_{q+1}\to i_{q+2},\dots,i_k\to i_{k+1}}|t_{j_{q+1},\dots,j_k}]\right)$$

$$\left.\otimes\prod_{1\le p\le k}(-1)^{j_p-i_p}y^{t_{j_p}}_{i_p\to i_{p+1}}[y^{-1}_{j_p-1,\dots,i_p}|t_{j_p}-t_{j_p-1,\dots,i_p}][y_{j_p+1,\dots,i_{p+1}-1}|t_{j_p+1,\dots,i_{p+1}-1}-t_{j_p}]\right\}$$

$$=\operatorname{INV}\left\{\sum_{1=i_1\le j_1<\cdots<i_{k+1}=d+1}\left(\right.\right.$$

$$-\sum_{1\le q\le k}\frac{(-1)^q}{t_{j_q}}[y^{-1}_{i_{q-1}\to i_q,\dots,i_1\to i_2}|-t_{j_{q-1},\dots,j_1}][y_{i_{q+1}\to i_{q+2},\dots,i_k\to i_{k+1}}|t_{j_{q+1},\dots,j_k}]$$

$$\left.+\sum_{1\le q\le k}\frac{(-1)^q}{t_{j_q}}[y^{-1}_{i_{q-1}\to i_q,\dots,i_1\to i_2}|t_{j_q}-t_{j_{q-1},\dots,j_1}]y^{t_{j_q}}_{1\to d+1}[y_{i_{q+1}\to i_{q+2},\dots,i_k\to i_{k+1}}|t_{j_{q+1},\dots,j_k}-t_{j_q}]\right)$$

$$\left.\otimes\prod_{1\le p\le k}(-1)^{j_p-i_p}y^{t_{j_p}}_{i_p\to i_{p+1}}[y^{-1}_{j_p-1,\dots,i_p}|t_{j_p}-t_{j_p-1,\dots,i_p}][y_{j_p+1,\dots,i_{p+1}-1}|t_{j_p+1,\dots,i_{p+1}-1}-t_{j_p}]\right\}. \tag{3.1.33}$$

This equals

$$=-\operatorname{INV}\left\{\sum_{1=i_0\le j_0<\cdots<i_{k+1}=d+1}\frac{(-1)^q}{t_{j_q}}[y^{-1}_{i_{q-1}\to i_q,\dots,i_0\to i_1}|-t_{j_{q-1},\dots,j_0}][y_{i_{q+1}\to i_{q+2},\dots,i_k\to i_{k+1}}|t_{j_{q+1},\dots,j_k}]\right.$$

$$\left.\otimes\prod_{0\le p\le k}(-1)^{j_p-i_p}y^{t_{j_p}}_{i_p\to i_{p+1}}[y^{-1}_{j_p-1,\dots,i_p}|t_{j_p}-t_{j_p-1,\dots,i_p}][y_{j_p+1,\dots,i_{p+1}-1}|t_{j_p+1,\dots,i_{p+1}-1}-t_{j_p}]\right\}$$

$$+\operatorname{INV}\left\{\sum_{1=i_0\le j_0<\cdots<i_{k+1}=d+1}\frac{(-1)^q}{t_{j_q}}[y^{-1}_{i_{q-1}\to i_q,\dots,i_0\to i_1}|t_{j_q}-t_{j_{q-1},\dots,j_0}]y^{t_{j_q}}_{1\to d+1}\right.$$

$$[y_{i_{q+1}\to i_{q+2},\dots,i_k\to i_{k+1}}|t_{j_{q+1},\dots,j_k}-t_{j_q}]$$

$$\left.\otimes\prod_{0\le p\le k}(-1)^{j_p-i_p}y^{t_{j_p}}_{i_p\to i_{p+1}}[y^{-1}_{j_p-1,\dots,i_p}|t_{j_p}-t_{j_p-1,\dots,i_p}][y_{j_p+1,\dots,i_{p+1}-1}|t_{j_p+1,\dots,i_{p+1}-1}-t_{j_p}]\right\} \tag{3.1.34}$$

which we write as $-\operatorname{INV}(T_3)+\operatorname{INV}(T_4)$. We note that

$$\Delta(B)=T_1,\qquad \Delta(C)=T_4,\qquad T_2=T_3. \tag{3.1.35}$$

This implies that $\mathrm{INV} \circ \Delta(A+B-C) = 0$, proving the claim. $\square$

3.2 Associated one-forms of multiple polylogarithms

3.2.1 Motivation

Recall Goncharov's definition of $\mathcal{B}_n(F)$ group as $\mathbb{Z}[\mathbb{P}^1_F]/R_n(F)$, where generators $[a]_n$ may be viewed as $\mathrm{Li}_n(a)$ and $R_n(F)$ is the subgroup generated by all functional relations between polylogarithms. These $\mathcal{B}_n(F)$ groups make up the Bloch complex (1.1.3)

$$\mathcal{B}_n(F) \xrightarrow{\delta_n} \mathcal{B}_{n-1}(F) \otimes F^\times \xrightarrow{\delta_{n-1}} \mathcal{B}_{n-1}(F) \otimes \bigwedge^2 F^\times \to \cdots \xrightarrow{\delta_2} \bigwedge^n F^\times$$

The differentials are

$$\begin{aligned} &\delta_n([a]) = [a] \otimes a, \quad \delta_2([a]) \otimes b = a \wedge (1-a) \wedge b \\ &\delta_k([a]) \otimes b = [a] \otimes a \wedge b, \quad \forall 2 < k < n \end{aligned} \tag{3.2.1}$$

Goncharov conjectured that the i-th cohomology group of this complex is rationally isomorphic to $H^i_{\mathcal{M}}(F, \mathbb{Q}(n))$.

In [12], Zickert considered lifted polylogarithms which are functions from $\widehat{\mathbb{C}}$ to $\mathbb{C}/\frac{(2\pi i)^n}{(n-1)!}\mathbb{Z}$ defined by

$$\widehat{\mathcal{L}}_n(u,v) = \sum_{r=0}^{n-1} \frac{(-1)^r}{r!} \mathrm{Li}_{n-r}(e^u) u^r - \frac{(-1)^n}{n!} u^{n-1} v \tag{3.2.2}$$

He then constructed groups $\widehat{\mathcal{B}}_n(\mathbb{C}) = \mathbb{Z}[\widehat{\mathbb{C}}]/\widehat{R}_n(\mathbb{C})$, where generators $[(u,v)]_n$ represent $\widehat{\mathcal{L}}_n(u,v)$ and $\widehat{R}_n(\mathbb{C})$ is generated by functional relations among $\widehat{\mathcal{L}}_n$. These $\widehat{\mathcal{B}}$ groups forms a chain complex $\widehat{\Gamma}(\mathbb{C}, n)$

$$\widehat{\mathcal{B}}_n(\mathbb{C}) \xrightarrow{\widehat{\delta}_n} \widehat{\mathcal{B}}_{n-1}(\mathbb{C}) \otimes \widehat{\mathbb{C}} \xrightarrow{\widehat{\delta}_{n-1}} \widehat{\mathcal{B}}_{n-2}(\mathbb{C}) \otimes \bigwedge^2 \widehat{\mathbb{C}} \to \cdots \xrightarrow{\widehat{\delta}_2} \bigwedge^n \widehat{\mathbb{C}} \tag{3.2.3}$$

where the differentials are

$$\begin{aligned}
&\widehat{\delta}_n([(u,v)]) = [(u,v)] \otimes u, \quad \widehat{\delta}_2([(u,v)]) \otimes a = u \wedge v \wedge a \\
&\widehat{\delta}_k([(u,v)]) \otimes a = [(u,v)] \otimes u \wedge a, \quad \forall 2 < k < n
\end{aligned} \tag{3.2.4}$$

Zickert conjectures that its i-th cohomology group is integrally isomorphic to $H^i_{\mathcal{M}}(\mathbb{C}, \mathbb{Z}(n))$. In particular, he argued that $\widehat{\mathcal{L}}_n$, viewed as a map from $\ker \widehat{\delta}_n$ to $\mathbb{C}/(2\pi i)^n\mathbb{Z}$, should correspond to the cycle map (see [**?**])

$$b_n : CH^n(\operatorname{Spec}\mathbb{C}, 2n-1) \to H^1_{\mathcal{D}}(\operatorname{Spec}\mathbb{C}, \mathbb{Z}(n)) \tag{3.2.5}$$

Since the Chow group $CH^n(X, 2n-i)$ isomorphic to $H^i_{\mathcal{M}}(X, \mathbb{Z}(n))$ for an algebraic manifold X (see [**?**]), and the Deligne cohomology $H^i_{\mathcal{D}}(\operatorname{Spec}\mathbb{C}, \mathbb{Z}(n))$ is isomorphic to $\mathbb{C}/(2\pi i)^n\mathbb{Z}$. Therefore b_n can be regarded as a map from $H^i_{\mathcal{M}}(\operatorname{Spec}\mathbb{C}, \mathbb{Z}(n))$ to $\mathbb{C}/(2\pi i)^n\mathbb{Z}$.

These lifted polylogarithms have well-defined differential one-forms in $\Omega^1(\widehat{\mathbb{C}})$:

$$w_n(u,v) = d\widehat{\mathcal{L}}_n = (-1)^n \frac{n-1}{n!} u^{n-2}(udv - vdu) \tag{3.2.6}$$

If we treat coordinates u, v as indeterminates in the polynomial ring $\mathbb{C}[u,v]$, then $w_n(u,v) \in \Omega^1_{\mathbb{C}[u,v]/\mathbb{C}}$. It is tempting to define lifted multiple polylogarithms $\widehat{\mathcal{L}}_{n_1,\cdots,n_d}$ such that

i. They are functions from $\widehat{S}_d(\mathbb{C})$ to $\mathbb{C}/\frac{(2\pi i)^n}{(n-1)!}\mathbb{Z}$, where $n = n_1 + \cdots + n_d$.

ii. They can be expressed as sums and products of multiple polylogarithms of lower weights.

iii. Their differentials $d\widehat{\mathcal{L}}_{n_1,\cdots,n_d}$ are in $\Omega^1_{\mathbb{C}[\{u_i, v_{j,k}\}]/\mathbb{C}}$.

Unfortunately, this cannot be done. In fact, Zickert defined $\widehat{\mathcal{L}}_{1,1}$, $\widehat{\mathcal{L}}_{2,1}$, $\widehat{\mathcal{L}}_{1,2}$, $\widehat{\mathcal{L}}_{1,1,1}$ (unpublished), but that the best one can do for $\widehat{\mathcal{L}}_{3,1}$ is such that

$$d\widehat{\mathcal{L}}_{3,1} = w_{3,1} - w_2(u_2, v_2)\widehat{\mathcal{L}}_2(u_{1,2}, v_{1,2}), \quad dw_{3,1} = w_2(u_2, v_2) \wedge w_2(u_{1,2}, v_{1,2}) \tag{3.2.7}$$

Here $u_{1,2}$ stands for $u_1 + u_2$. Note that $w_{3,1}$ is neither exact (not the differential of $\widehat{\mathcal{L}}_{3,1}$) nor closed. This suggests a Hodge structure hidden behind these lifted multiple polylogarithms, similar to Zhao's description of variation of Hodge structures of multiple polylogarithms. We will tackle this problem in Chapter 4.

These one-forms $w_{n_1,\cdots,n_d}$ can be obtained from the symbols of the multiple polylogarithm $\mathrm{Li}_{n_1,\cdots,n_d}$ [2]. By first applying Gangl's projection map P in Definition 2.2.17 and then the map

$$a_1 \otimes \cdots \otimes a_n \to \frac{(-1)^{n+1}}{(n-1)!} a_2 \cdots a_n da_1 \tag{3.2.8}$$

These one-forms have nice combinatorial properties. We will discuss them in the next section and later on in Chapter 4.

3.2.2 One-forms

Definition 3.2.1. Let H be a connected graded Hopf algebra. Consider the *one-forms* map $w_f : T^*H_1 \to \Omega^1_{\mathbb{Q}[H_1]/\mathbb{Q}}$

$$w_f(a_1 \otimes \cdots \otimes a_n) = \frac{(-1)^{n+1}}{n!} \sum_{1 \le i \le n} (-1)^{i-1} \binom{n-1}{i-1} a_1 \cdots da_i \cdots a_n \tag{3.2.9}$$

Theorem 3.2.2. w_f is equal to the composition of P in Definition 2.2.17 and map in (3.2.8).

Proof. The proof can be found in [2] Lemma 4.3 with Π as P. $\square$

Definition 3.2.3. The one form map $w : \mathbb{H}^{\text{Symb}} \to \Omega_{\mathbb{Q}[u_i, v_{j,k}]/\mathbb{Q}}$ is defined to be the composite of the symbol map $\Delta_{1,\cdots,1} : \mathbb{H}^{\text{Symb}} \to T^*\mathbb{H}_1^{\text{Symb}}$, $w_f : T^*\mathbb{H}_1^{\text{Symb}} \to \Omega_{\mathbb{Q}[\mathbb{H}_1^{\text{Symb}}]/\mathbb{Q}}$, and $w_s : \Omega_{\mathbb{Q}[\mathbb{H}_1^{\text{Symb}}]/\mathbb{Q}} \to \Omega_{\mathbb{Q}[\{u_i, v_{j,k}\}]/\mathbb{Q}}$. Where w_s substitute $[x_i]_0$ into u_i and $[x_{j\to k+1}]_1$ into $-v_{j,k}$.

Example 3.2.4.

$$w([x_1]_n) = \frac{(-1)^n}{n!} u_1^{n-2}(u_1 dv_1 - v_1 du_1), \quad n \geq 2 \tag{3.2.10}$$

$$\begin{aligned} w([x_1, x_2]_{1,1}) = &-\frac{1}{2}u_1 dv_{1,2} + \frac{1}{2}v_{1,2} du_1 + \frac{1}{2}v_1 dv_{1,2} - \frac{1}{2}v_2 dv_{1,2} \\ &- \frac{1}{2}v_{1,2} dv_1 + \frac{1}{2}v_{1,2} dv_2 - \frac{1}{2}v_1 dv_2 + \frac{1}{2}v_2 dv_1 \end{aligned} \tag{3.2.11}$$

Note that(3.2.10) corresponds to (3.2.6) up to a constant $(n-1)$.

3.3 Free contraction Hopf algebras

We will construct several different Hopf algebras at the end of this section, and justifying the Hopf algebra structure of each one individually would require significant effort. Instead, we introduced the notion of a contraction system, which captures the essence of the coproduct 3.1.2. Consider maps $(i_1, i_2, \cdots, i_{d+1})$ that contracts variables

$$\begin{aligned} S_n(\mathbb{C}) &\xrightarrow{(i_1, i_2, \cdots, i_{d+1})} S_d(\mathbb{C}) \\ (y_1, y_2, \cdots, y_n) &\mapsto \left(\prod_{i_1 \leq r < i_2} y_r, \prod_{i_2 \leq r < i_3} y_r, \cdots, \prod_{i_d \leq r < i_{d+1}} y_r \right) \end{aligned} \tag{3.3.1}$$

Then the pullback

$$(i_1, i_2, \cdots, i_{d+1})^* \operatorname{Li}_{n_1,\cdots,n_d}(y_1, \cdots, y_n) = \operatorname{Li}_{n_1,\cdots,n_d}(y_{i_1\to i_2}, \cdots, y_{i_d \to i_{d+1}}) \tag{3.3.2}$$

appears in the second term of the tensor products in the coproduct formula 3.1.2.

3.3.1 Contraction System

A contraction system is a category that formalizes the contraction property described in (3.3.2). In fact, it is equivalent to a semisimplical category with $X_1 = \{*\}$. However, we retain the term "contraction system" because it is more suggestive of the context of our work. Hopf algebras in Definition 3.3.6 are called contraction algebras due to this.

Let's denote $\mathscr{C}$ as the category where

- The objects are sets of the form $[n] = \{0, 1, \cdots, n\}$ for $n \geq 1$.
- A morphism from $[n]$ to $[d]$ is given by a tuple $\mathbf{i} = (i_1, \cdots, i_{d+1})$, $1 \leq i_1 < \cdots < i_{d+1} \leq n+1$.
- The identity morphism for each object $[n]$ is $\mathbf{1} = (1, 2, \cdots, n+1)$
- and composition of $\mathbf{j} = (j_1, \cdots, j_{k+1}) : [d] \to [k]$ and $\mathbf{i}$ is $\mathbf{j} \circ \mathbf{i} = (i_{j_1}, \cdots, i_{j_{k+1}})$.

Example 3.3.1. Let $\mathcal{X} = \bigcup_d \mathcal{X}^d$ be the set of continuous products of x symbols where $\mathcal{X}^d = \{(x_{i_1 \to i_2}, \cdots, x_{i_d \to i_{d+1}})\}$. Then $\mathcal{X}$ forms a contraction system with contractions

$$\mathcal{X}^d \xrightarrow{(j_1, \cdots, j_{k+1})} \mathcal{X}^k, \quad (x_{i_1 \to i_2}, \cdots, x_{i_d \to i_{d+1}}) \mapsto (x_{i_{j_1} \to i_{j_2}}, \cdots, x_{i_{j_k} \to i_{j_{k+1}}})$$

With this notation, (3.1.2) becomes

$$\Delta([\mathbf{y}|\mathbf{t}]) = \sum [(i_1, \cdots, i_{d+1})\mathbf{y}|t_{j_1}, \ldots, t_{j_k}] \bigotimes \prod_{\alpha=0}^{k} (-1)^{j_\alpha - i_\alpha} \exp([(i_\alpha, i_{\alpha+1})\mathbf{y}]_0 t_{j_\alpha})$$

$$[((i_\alpha, i_\alpha + 1, \cdots, j_\alpha)\mathbf{y})^{-1}|t_{j_\alpha} - t_{j_\alpha - 1}, \ldots, t_{j_\alpha} - t_{i_\alpha}]$$

$$[(j_\alpha + 1, j_\alpha + 2, \cdots, i_{\alpha+1})\mathbf{y}|t_{j_\alpha+1} - t_{j_\alpha}, \ldots, t_{i_{\alpha+1}-1} - t_{j_\alpha}]. \tag{3.3.3}$$

The sum is over all instances of $1 = i_0 \leq j_0 < i_1 \leq j_1 < \cdots < i_k \leq j_k < i_{k+1} = d + 1$.

Example 3.3.2. We can turn the maps in Example 3.3.1 into maps between schemes

$$\left\{ S_n = \operatorname{Spec} \mathbb{Z} \left[x_i^{\pm 1}, \left(\prod_{j \leq r < k} x_r - 1 \right)^{-1} \right] \right\}$$

where $S_n \xrightarrow{(i_1, \cdots, i_{d+1})} S_d$ is given by $x_p \mapsto \prod\limits_{i_p \leq r < i_{p+1}} x_r$.

We would like to interpret the sets in both examples as functors, this leads to a formal definition of a contraction system.

Definition 3.3.3. A *contraction system* in $\mathscr{D}$ is defined as a functor from $\mathscr{C}$ to $\mathscr{D}$. A *morphism between contraction systems* is a natural transformation.

Example 3.3.4. Given a field F, all morphisms between contraction systems $\mathcal{X} \to \{S_n(F)\}$ are in one-to-one correspondence with tuples $\{(a_i)_{i \in \mathbb{Z}_{>0}}, a_i \in F\}$ determined by $\{x_i \to a_i\}_{i \in \mathbb{Z}_{>0}}$

Example 3.3.5. Suppose F is a field and $\pi : E \to F^*$ is a torsion free extension of F^* by $\mathbb{Z}$. Similar to the construction of $\widehat{S}_d(\mathbb{C})$ (where $\pi = \exp : \mathbb{C} \to \mathbb{C}^*$), we can construct a contraction system $\{\widehat{S}_n(F)\}$ as

$$\widehat{S}_n(F) = \left\{ (\{u_i\}_{1 \leq i \leq n}, \{v_{j,k}\}_{1 \leq j \leq k \leq n}) \in E^{n + \binom{n+1}{2}} \middle| \pi \left(\textstyle\sum_{r=j}^{k} u_r \right) + \pi(v_{j,k}) = 1, \forall j \leq k \right\} \tag{3.3.4}$$

and the contraction maps are

$$(i_1, \cdots, i_{d+1}) \left(\{u_p\}_p, \{v_{j,k}\}_{j \le k}\right) = \left(\left\{\sum_{r=i_q}^{i_{q+1}-1} u_r\right\}_q, \left\{v_{i_j, i_{k+1}-1}\right\}_{j \le k}\right) \tag{3.3.5}$$

3.3.2 Free contraction Hopf algebra

In this subsection, we want to construct $\overline{\mathbb{H}}$ and $\mathbb{H}$ as functors from the category of contraction systems to the category of graded Hopf algebras.

Definition 3.3.6. [2] For any contraction system $\mathcal{A}$, we define $\overline{\mathbb{H}}(\mathcal{A})$ to be the free algebra generated by regular symbols $[\alpha]_{n_1,\cdots,n_d}$ and inverted symbols $[\alpha^{-1}]_{n_d,\cdots,n_1}$, where $\alpha \in \mathcal{A}^d, \mathbf{n} = (n_1, \cdots, n_d) \in \mathbb{Z}^d_{>0}$ and $[\alpha]_0, [\alpha^{-1}]_0$, $\alpha \in \mathcal{A}^1$ modulo relations

$$[(i_1, i_2)\alpha]_0 + [(i_2, i_3)\alpha]_0 = [(i_1, i_3)\alpha]_0, \forall \alpha \in \mathcal{A}^d, d \ge 2, \quad [\alpha^{-1}]_0 = -[\alpha]_0$$

Here d is referred to as the depth. We also denote $\mathbb{H}(\mathcal{A})$ as the subalgebra of $\overline{\mathbb{H}}(\mathcal{A})$ generated solely by regular symbols $[\alpha]_{n_1,\cdots,n_d}$, $[\alpha]_0$.

Similar to (3.1.8), we can also define INV : $\overline{\mathbb{H}}(\mathcal{A}) \to \overline{\mathbb{H}}(\mathcal{A})$ which fixes the regular symbols and acts on inverted symbols inductively as

$$\begin{aligned}
&\mathrm{INV}([\mathbf{y}^{-1}| - t_d, \cdots, -t_1]) \\
&= \sum_{j=0}^{d-1} (-1)^{d-1+j}\, \mathrm{INV}([((1, \cdots, j+1)\mathbf{y})^{-1}| - t_j, \cdots, -t_1])[(j+1, \cdots, d+1)\mathbf{y}|t_{j+1}, \cdots, t_d] \\
&+ \sum_{j=1}^{d} \frac{(-1)^{d-1+j}}{t_j}\, \mathrm{INV}([((1, \cdots, j)\mathbf{y})^{-1}| - t_{j-1}, \cdots, -t_1])[(j+1, \cdots, d+1)\mathbf{y}|t_{j+1}, \cdots, t_d] \\
&+ \sum_{j=1}^{d} \Bigg(\frac{(-1)^{d+j}}{t_j}\, \mathrm{INV}([((1, \cdots, j)\mathbf{y})^{-1}|t_j - t_{j-1}, \cdots, t_j - t_1]) \\
&\qquad \exp([(1, d+1)\mathbf{y}]_0 t_j)[(j+1, \cdots, d+1)\mathbf{y}|t_{j+1} - t_j, \cdots, t_d - t_j] \Bigg)
\end{aligned} \tag{3.3.6}$$

Proposition 3.3.7. The coproduct Δ given by (3.3.3) and

$$\Delta([\alpha^{\pm}]_0) = 1 \otimes [\alpha]_0 + [\alpha^{\pm}]_0 \otimes 1, \quad \Delta([\alpha]_1^{\pm}) = 1 \otimes [\alpha^{\pm}]_1 + [\alpha^{\pm}]_1 \otimes 1$$

on any $\overline{\mathbb{H}}(\mathcal{A})$ and $\mathbb{H}(\mathcal{A})$ define Hopf algebra structures.

Proof. The proof is straightforward. See [2], Theorem 2.15. $\square$

Example 3.3.8. With $\mathcal{X}$ defined in Example 3.3.1, it is not hard to see that $\overline{\mathbb{H}}(\mathcal{X}) = \overline{\mathbb{H}}^{\mathrm{Symb}}$, $\mathbb{H}(\mathcal{X}) = \mathbb{H}^{\mathrm{Symb}}$.

We call these $\overline{\mathbb{H}}(\mathcal{A})$ and $\mathbb{H}(\mathcal{A})$ *free contraction Hopf algebras.* And we have the following proposition.

Proposition 3.3.9. $\overline{\mathbb{H}}$, $\mathbb{H}$ are functors from the category of contraction systems to the category of graded Hopf algebras.

Example 3.3.10. Suppose $\mathcal{A}$ is the contraction system $\{S_n(F)\}$, then the corresponding free contraction Hopf algebras are denoted by $\overline{\mathbb{H}}^{\mathrm{Symb}}(F)$, $\mathbb{H}^{\mathrm{Symb}}(F)$. The corresponding free contraction Hopf algebra of $\{\widehat{S}_n(F)\}$ in Example 3.3.5 is denoted $\widehat{\mathbb{H}}_E^{\mathrm{Symb}}(F)$ or $\widehat{\mathbb{H}}^{\mathrm{Symb}}(F)$ for short. Specifically, if $E = F = \mathbb{C}$ and $\pi : \mathbb{C} \to \mathbb{C}^*$ is the exponential map, we get the free contraction Hopf algebra as $\widehat{\mathbb{H}}^{\mathrm{Symb}}(\mathbb{C})$.

More generally, we can construct sheaves of Hopf algebras over manifolds.

Example 3.3.11. Let M be a smooth complex manifold and U be an open subset of M, then holomorphic maps from U to $\widehat{S}_n(\mathbb{C})$ forms a contraction system where the contraction maps $\Omega^0(U, \widehat{S}_n(\mathbb{C})) \to \Omega^0(U, \widehat{S}_d(\mathbb{C}))$ are simply induced by contraction maps $\widehat{S}_n(\mathbb{C}) \to \widehat{S}_d(\mathbb{C})$. This way we obtain a sheaf $\widehat{\mathbb{H}}_M^{\mathrm{Symb}}$ of Hopf algebras on M.

3.4 Motivic complex

In this section, we assume $0 \to \mathbb{Z} \to E \xrightarrow{\pi} F^* \to 0$ is a torsion free $\mathbb{Z}$ extension, and construct candidates for motivic complexes in Question 1.1.1 ii. and Question 1.1.2 iii.. And we formulate the conjecture that relates motivic cohomology and singular cohomology through our construction.

3.4.1 The $\mathbb{L}^{\mathrm{Symb}}$ complex

Recall that according to Definition 2.2.14, any connected graded Hopf algebra modulo products defines a Lie coalgebra, it is natural to define Lie coalgebras

$$\begin{aligned}
\mathbb{L}^{\mathrm{Symb}} &:= \mathbb{H}^{\mathrm{Symb}}/(\mathbb{H}^{\mathrm{Symb}}_{>0} \cdot \mathbb{H}^{\mathrm{Symb}}_{>0}) \\
\widehat{\mathbb{L}}^{\mathrm{Symb}}(F) &:= \widehat{\mathbb{H}}^{\mathrm{Symb}}(F)/\left(\widehat{\mathbb{H}}^{\mathrm{Symb}}(F)_{>0} \cdot \widehat{\mathbb{H}}^{\mathrm{Symb}}(F)_{>0}\right) \qquad (3.4.1)\\
\widehat{\mathbb{L}}^{\mathrm{Symb}}_M &:= \widehat{\mathbb{H}}^{\mathrm{Symb}}_M/\left({\widehat{\mathbb{H}}^{\mathrm{Symb}}_M}_{>0} \cdot {\widehat{\mathbb{H}}^{\mathrm{Symb}}_M}_{>0}\right)
\end{aligned}$$

The one form map $\mathbb{H}^{\mathrm{Symb}} \to \Omega^1_{\mathbb{Q}[\{u_i,v_{j,k}\}]/\mathbb{Q}}$ induces a chain map from its Chevalley-Eilenberg complex $\bigwedge^* \mathbb{L}^{\mathrm{Symb}}$ to the de Rham complex $\Omega^*_{\mathbb{Q}[\{u_i,v_{j,k}\}]/\mathbb{Q}}$.

Remark 3.4.1. Even though $\mathbb{H}^{\mathrm{Symb}}$, $\widehat{\mathbb{H}}^{\mathrm{Symb}}(F)$ and $\widehat{\mathbb{H}}^{\mathrm{Symb}}_M$ are defined over $\mathbb{Q}$, but $\mathbb{L}^{\mathrm{Symb}}$, $\widehat{\mathbb{L}}^{\mathrm{Symb}}(F)$ and $\widehat{\mathbb{L}}^{\mathrm{Symb}}_M$ are defined over $\mathbb{Z}$.

Theorem 3.4.2. [2] The one-form map induces a chain map from $\bigwedge^* \mathbb{L}^{\mathrm{Symb}}$ to $\Omega^*_{\mathbb{Q}[\{u_i,v_{j,k}\}]/\mathbb{Q}}$, i.e. the following diagram commutes

$$\begin{tikzcd}
\mathbb{L}^{\mathrm{Symb}} \arrow[r, "\delta"] \arrow[d, "w"] & \bigwedge^2 \mathbb{L}^{\mathrm{Symb}} \arrow[r, "1\wedge\delta-\delta\wedge 1"] \arrow[d, "w\wedge w"] & \bigwedge^3 \mathbb{L}^{\mathrm{Symb}} \arrow[r] \arrow[d, "w\wedge w\wedge w"] & \cdots \\
\Omega^1 \arrow[r, "d"] & \Omega^2 \arrow[r, "d"] & \Omega^3 \arrow[r, "d"] & \cdots
\end{tikzcd}$$

Proof. First we show that w_f defined in (3.2.9) induces a chain map $w_f : \bigwedge^* L(T(H_1)) \to \Omega^*$, where $L(T(H_1)) = \frac{T(H_1)}{T(H_1)_{>0} \cdot T(H_1)_{>0}}$. This is Proposition 4.4 in [2].

On the other hand, $\Delta_{1,\cdots,1}$ induces a chain map $L(H)$ to $L(T(H_1))$ by Proposition 2.2.20, where $L(H) = \frac{H}{H_{>0} \cdot H_{>0}}$. Composing both chain maps, we obtain a chain map from $\bigwedge^* \mathbb{L}^{\mathrm{Symb}}$ to $\Omega^*_{\mathbb{Q}[\{u_i, v_{j,k}\}]/\mathbb{Q}}$. $\square$

3.4.2 Main conjectures

In [12], Zickert defined the lifted Bloch complex $\widehat{\Gamma}(F, n)$

$$\widehat{\mathcal{B}}_n(F) \xrightarrow{\widehat{\delta}_1} \widehat{\mathcal{B}}_{n-1}(F) \otimes E \xrightarrow{\widehat{\delta}_2} \widehat{\mathcal{B}}_{n-2}(F) \otimes \bigwedge^2 E \to \cdots \to \widehat{\mathcal{B}}_2(F) \otimes \bigwedge^{n-2} E \xrightarrow{\widehat{\delta}_{n-1}} \bigwedge^n E \tag{3.4.2}$$

with differentials given by 3.2.4. Assuming $\pi : E \to F^*$ is an $\mathbb{Z}$ extension. He conjectured that the i-th cohomology group of this complex is integrally isomorphic to $H^i_{\mathcal{M}}(F, \mathbb{Z}(n))$.

We wish to generalize the Bloch complex (1.1.3) and the lifted Bloch complex (3.4.2) to include generators representing multiple polylogarithms and lifted multiple polylogarithms. Similar to the definition of $\mathcal{B}$ and $\widehat{\mathcal{B}}$ groups, we define $\mathbb{L}_n(F)$ and $\widehat{\mathbb{L}}_n(F)$ to be $\mathbb{L}_n^{\mathrm{Symb}}(F)/R_n(F)$ and $\widehat{\mathbb{L}}_n^{\mathrm{Symb}}(F)/\widehat{R}_n(F)$ respectively. Where $R_n(F)$ is inductively defined similar to Goncharov [7], which can be found in [1], section 3. And $\widehat{R}_n(F)$ is yet unknown, but it should be constructed similar to that in [12], and it is expected to be generated by all functional relations between lifted multiple polylogarithms of weight n.

Their Chevalley-Eilenberg complexes $\bigwedge^* \mathbb{L}(F)$ and $\bigwedge^* \widehat{\mathbb{L}}(F)$ will be referred to as the motivic complex and lifted motivic complex. We conjecture that they compute rational and integral motivic cohomology, respectively.

Conjecture 3.4.3.

$$H^i\Big((\bigwedge^* \mathbb{L}(F))_n \Big)_{\mathbb{Q}} \cong H^i_{\mathcal{M}}(F, \mathbb{Q}(n))_{\mathbb{Q}}, \quad H^i\Big(\Big(\bigwedge^* \widehat{\mathbb{L}}(F)\Big)_n \Big) \cong H^i_{\mathcal{M}}(F, \mathbb{Z}(n))$$

We use subscript $\mathbb{Q}$ in the first isomorphism to indicate that this only conjectured to be true rationally.

$\widehat{\mathbb{L}}^{\mathrm{Symb}}_M$ also defines a sheaf of Lie coalgebras over a complex manifold M. Theorem 3.4.2 induces a sheaf map $\bigwedge^* \widehat{\mathbb{L}}^{\mathrm{Symb}}_M \to \Omega^*_M$ defined locally by $\widehat{\mathbb{L}}^{\mathrm{Symb}}_M(U) \to \Omega^1_M(U)$ as pulling back of one-form map w. For example, for $f = (f_1, f_2, f_3, f_4, f_5) \colon U \to \widehat{S}_2$, that

$$\begin{aligned} w([f]_{1,1}) = \frac{1}{2}(&-f_1^* u_1 f_5^* dv_{1,2} + f_5^* v_{1,2} f_1^* du_1 + f_3^* v_1 f_5^* dv_{1,2} \\ &- f_4^* v_2 f_5^* dv_{1,2} - f_5^* v_{1,2} f_3^* dv_1 + f_5^* v_{1,2} f_4^* dv_2 - f_3^* v_1 f_4^* dv_2 + f_4^* v_2 f_3^* dv_1) \in \Omega^1_M(U). \end{aligned} \tag{3.4.3}$$

Assume R_M is the subsheaf of $\widehat{\mathbb{L}}^{\mathrm{Symb}}_M$ generated by functional relations of lifted multiple polylogarithms pullback to M. Then $\bigwedge^* \widehat{\mathbb{L}}_M \to \Omega^*_M$ is well defined. Since one forms are differentials of lifted multiple polylogarithms modulo products of lower weight functions, and the differential of any functional relation should be zero.

Conjecture 3.4.4. [2] If M is an algebraic manifold, we conjecture the following diagram commutes

$$\begin{tikzcd} H^i(M, (\bigwedge^* \widehat{\mathbb{L}}_M)_n) \arrow[r] & H^i(M, \Omega^*_M) \arrow[d, "\cong"] \\ H^i_{\mathcal{M}}(M, \mathbb{Z}(n)) \arrow[r] \arrow[u, dashed] & H^i(M, \mathbb{C}) \end{tikzcd}$$

Recall from Definition 2.2.15 that $(\bigwedge^* \widehat{\mathbb{L}}_M)_n$ is the degree n part of the complex. The top arrow is induced by $\bigwedge^\bullet \widehat{\mathbb{L}}_M \to \Omega^\bullet$, the bottom arrow is the realization functor from integral motivic cohomology to singular cohomology, the right arrow is the isomorphism between de Rham cohomology and singular cohomology [14], and the left arrow is yet unclear and requires further investigation.

Chapter 4: Variation of Mixed Hodge Structures of Multiple Polylogarithms

Deligne and Beilinson first constructed the variation matrix

$$L(z) = \begin{bmatrix} 1 & & & & & \\ \mathrm{Li}_1(z) & 1 & & & & \\ \mathrm{Li}_2(z) & \log(z) & 1 & & & \\ \mathrm{Li}_3(z) & \frac{1}{2}\log^2(z) & \log(z) & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ \mathrm{Li}_n(z) & \frac{1}{(n-1)!}\log^{n-1}(z) & \frac{1}{(n-2)!}\log^{n-2}(z) & \cdots & \log(z) & 1 \end{bmatrix} \tag{4.0.1}$$

as the fundamental solution to the linear partial differential equation

$$dL(z) = \begin{bmatrix} 0 & & & & & \\ \frac{dz}{1-z} & 0 & & & & \\ & \frac{dz}{z} & 0 & & & \\ & & \frac{dz}{z} & 0 & & \\ & & & \ddots & \ddots & \\ & & & & \frac{dz}{z} & 0 \end{bmatrix} L(z) \tag{4.0.2}$$

They computed the monodromy matrices for $L(z)\tau(2\pi i)$ which are

$$M_{\sigma_0} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & 1 & 1 & & & \\ & \frac{1}{2} & 1 & 1 & & \\ & \vdots & \ddots & \ddots & \ddots & \\ & \frac{1}{(n-1)!} & \cdots & \frac{1}{2} & 1 & 1 \end{bmatrix}, \quad M_{\sigma_1} = \begin{bmatrix} 1 & & & & & \\ -1 & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix} \tag{4.0.3}$$

Here σ_0, σ_1 are loops around 0, 1, generating $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\})$, and

$$\tau(2\pi i) = \begin{bmatrix} 1 & & & & & \\ & 2\pi i & & & & \\ & & (2\pi i)^2 & & & \\ & & & (2\pi i)^3 & & \\ & & & & \ddots & \\ & & & & & (2\pi i)^n \end{bmatrix}$$

In this chapter, we will construct variation matrices for multiple polylogarithms, describe variations of mixed Hodge structures with it. This has been done by Zhao [9], but only for multiple logarithms. And at the last section, we discuss two other potential applications of variation matrices.

4.1 Variation matrix

Goncharov's coproduct (2.3.13) naturally defines a matrix satisfying Theorem 4.1.3, which we will refer to as the variation matrix. This matrix is a lower-triangular unipotent square matrix, with rows and columns determined by Goncharov's coproduct formula (2.3.13) on iterated integrals. Goncharov used this concept to describe a variation of mixed Hodge structures on the torsor of path in $\mathbb{C} - \{a_1, \cdots, a_n\}$ (see [21], Section 5).

In this section, we shall first prove that the variation matrix behaves as a group-like element under the coproduct. We also derive a matrix of symbols by repeatedly applying the coproduct, and a matrix of one-forms by further apply the one-form map w to these symbols.

4.1.1 Variation matrix for iterated integrals

Definition 4.1.1. Suppose $\{a_i\}_{i=0}^{n+1} \subseteq S$, the rows and columns of Goncharov's variation matrix $V^{\widetilde{\mathcal{I}}}$ are parameterized by tuples

$$\mathbf{i} = (i_0, \cdots, i_{n+1}), \qquad 0 = i_0 < i_1 < \cdots < i_d < i_{d+1} = n+1$$

and entries in $\widetilde{\mathcal{I}}(S)$. Specifically, the $(\mathbf{i}, \mathbf{j})$ entry of V is defined as

$$V^{\widetilde{\mathcal{I}}}_{\mathbf{i},\mathbf{j}} = \begin{cases} \prod_{p=0}^{m} I(a_{i_{k_p}}; a_{i_{k_p+1}}, \cdots ; a_{i_{k_{p+1}}}), \text{if } \mathbf{j} = (i_{k_0}, \cdots, i_{k_{m+1}}) \text{ is a subsequence of } \mathbf{i} \\ 0, \text{otherwise} \end{cases} \tag{4.1.1}$$

Remark 4.1.2. Note that here we did not specify the order of rows and columns of the variation matrix. In fact, we can choose any ordering of the indices as long as we make sure that if $\mathbf{j}$ is a subsequence of $\mathbf{i}$, then the $\mathbf{j}$-th row/column is before the $\mathbf{i}$-th row/column.

We are now ready to state the group-like hehavior of the variation matrix.

Theorem 4.1.3. We have $\Delta(V^{\widetilde{\mathcal{I}}})^T = (V^{\widetilde{\mathcal{I}}})^T \otimes (V^{\widetilde{\mathcal{I}}})^T$. Here $\cdot^T$ means matrix transpose.

Proof. This is a direct corollary of Theorem 4.1.8. $\square$

We say that $V^{\widetilde{\mathcal{I}}}$ is a profinite matrix because for a fixed $\mathbf{i} = (i_0, \cdots, i_{n+1})$ we can define the variation (square) matrix for $I(a_{i_0}, \cdots, a_{i_{n+1}})$ to be the submatrix with rows and columns indexed by all indices $\mathbf{j} = (i_{k_0}, \cdots, i_{k_{m+1}})$ with $k_0 = 0$, $k_{m+1} = n+1$.

Example 4.1.4. The variation matrix for $I(a_0; a_1, a_2; a_3)$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ I(a_0; a_1; a_3) & 1 & 0 & 0 \\ I(a_0; a_2; a_3) & 0 & 1 & 0 \\ I(a_0; a_1, a_2; a_3) & I(a_1; a_2; a_3) & I(a_0; a_1; a_2) & 1 \end{bmatrix} \tag{4.1.2}$$

4.1.2 Variation matrix for free contraction Hopf algebras

We would like to generalize the concept of variation matrix by defining it for a free contraction Hopf algebra H, equipped with a total graded ordering $\prec$ on its generators. This generalization allows us to state and derive results about the variation matrix in a more abstract setting, making it easier to explore its properties.

Definition 4.1.5. Suppose H is a free contraction Hopf algebra with $\mathfrak{W}$ being the set of generators, and there is a total graded ordering $\prec$ on $\mathfrak{W}$. The *variation matrix* is a matrix V^H where

- the first column is all of its generators in order.

- the rest of the matrix are determined by the coproduct formula

$$\Delta(v) = \sum_{w \in \mathfrak{W}} w \otimes V^H_{v,w}$$

We call the entry of $V^H_{v,w}$ the *complementary entry* of w with respect to v. If in addition $\mathfrak{W}$ is isomorphic to $(\mathbb{Z}^{\infty}_{\geq 0}, \prec)$ in 3.1.3 as a totally ordered graded countable set. Then we can parameterize the rows and columns of V^H by indices in $\mathbb{Z}^{\infty}_{\geq 0}$.

Remark 4.1.6. Note that $V^H_{v,w} = 0$ if $\dim(v) \neq \dim(w)$ or if $v \prec w$, so V^H is a lower right triangular unipotent matrix, and there are only finitely many entries in each row and column if the coproduct formula is finite.

Using this definition, the variation matrix from Definition 4.1.1 becomes a variation matrix with $H = \widetilde{I}(S)$.

While writing down the entire variation matrix is impractical due to it profinite nature, it is feasible to express its submatrices truncated at specific entries, allowing us to study the matrix while retaining essential information about its structure.

Definition 4.1.7. The variation matrix of an element v is the submatrix of V^H with the rows whose first entry appears in the coproduct of $\Delta(v)$, and columns of the same indices as rows.

Since the definition of the variation matrix relies on the coproduct, the group-like property is now obtained automatically. This simplifies our analysis, as the matrix inherently satisfies this key structural property.

Theorem 4.1.8. The variation matrix satisfies $\Delta(V^H)^T = (V^H)^T \otimes (V^H)^T$

Proof. The proof simply uses coassociativity of the coproduct. First, note that

$$\Delta^2(v) = \sum_w w \otimes \Delta(V^H_{v,w})$$

On the other hand

$$\begin{aligned}
\Delta^2(v) &= \sum_w \Delta(w) \otimes V^H_{v,w} \\
&= \sum_w \left(\sum_u u \otimes V^H_{w,u} \right) \otimes V^H_{v,w} \\
&= \sum_{w,u} u \otimes V^H_{w,u} \otimes V^H_{v,w} \\
&= \sum_{u,w} w \otimes V^H_{u,w} \otimes V^H_{v,u} \\
&= \sum_w w \otimes \left(\sum_u V^H_{u,w} \otimes V^H_{v,u} \right)
\end{aligned}$$

Comparing both we have

$$\Delta(V^H_{v,w}) = \sum_u V^H_{u,w} \otimes V^H_{v,u} \tag{4.1.3}$$

This concludes the theorem. □

Recall the Definition 3.1.15. $H = \mathbb{I}^{\text{Symb}}$ produces a variation matrix $V^{\mathbb{I}}$. Apply the Hopf algebra morphism Φ and map INV, we will subsequently derive variation matrices $V^{\overline{\mathbb{H}}} = \Phi(V^{\mathbb{I}})$ and $V^{\mathbb{H}} = \text{INV}(V^{\overline{\mathbb{H}}})$. Each of these three variation matrices can be useful in different scenarios, depending on the specific properties or structures we wish to emphasize or explore.

Example 4.1.9. The $V^{\mathbb{I}}$ variation matrix of $I(0; a_1, 0, a_2; a_3)$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ I(0; a_2; a_3) & 1 & 0 & 0 & 0 & 0 \\ I(0; a_1; a_3) & 0 & 1 & 0 & 0 & 0 \\ I(0; a_1, a_2; a_3) & I(0; a_1; a_2) & I(a_1; a_2; a_3) & 1 & 0 & 0 \\ I(0; a_1, 0; a_3) & 0 & I(a_1; 0; a_3) & 0 & 1 & 0 \\ I(0; a_1, 0, a_2; a_3) & I(0; a_1, 0; a_2) & I(a_1; 0, a_2; a_3) & I(a_1; 0; a_2) & I(0; a_2; a_3) & 1 \end{bmatrix} \tag{4.1.4}$$

Example 4.1.10. The $V^{\overline{\mathbb{H}}}$ variation matrix of $[x_1, x_2]_{2,1}$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ [x_2]_1 & 1 & 0 & 0 & 0 & 0 \\ [x_1x_2]_1 & 0 & 1 & 0 & 0 & 0 \\ [x_1, x_2]_{1,1} & [x_1]_1 & [x_2]_1 - [x_1^{-1}]_1 & 1 & 0 & 0 \\ [x_1x_2]_2 & 0 & [x_1x_2]_0 & 0 & 1 & 0 \\ [x_2, x_2]_{2,1} & [x_1]_2 & [x_1^{-1}]_2 - [x_2]_2 + [x_2]_1[x_1x_2]_0 & [x_1]_0 & [x_2]_1 & 1 \end{bmatrix} \tag{4.1.5}$$

The $V^{\mathbb{H}}$ variation matrix of $[x_1, x_2]_{2,1}$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ [x_2]_1 & 1 & 0 & 0 & 0 & 0 \\ [x_1x_2]_1 & 0 & 1 & 0 & 0 & 0 \\ [x_1, x_2]_{1,1} & [x_1]_1 & [x_2]_1 - [x_1]_1 - [x_1]_0 & 1 & 0 & 0 \\ [x_1x_2]_2 & 0 & [x_1x_2]_0 & 0 & 1 & 0 \\ [x_2, x_2]_{2,1} & [x_1]_2 & -[x_1]_2 - [x_2]_2 - \frac{1}{2}[x_1]_0^2 + [x_2]_1[x_1x_2]_0 & [x_1]_0 & [x_2]_1 & 1 \end{bmatrix} \tag{4.1.6}$$

To better prepare for the monodromy computations in Chapter 5, we will now discuss the structure of $V^{\mathbb{I}}$ in much greater detail. To streamline this discussion, we will introduce some shorthand notations that will make the expressions more manageable and concise.

Definition 4.1.11. Inspired by Zhao [9]. For a word $w = \sigma_{j_1} \cdots \sigma_{j_m}$ we define $I^w(a_{i_0}; a_{i_1}, \cdots, a_{i_n}; a_{i_{n+1}})$ to be 0 if $(j_1, \cdots, j_m)$ is a not subsequence of $(i_1, \cdots, i_n)$ and otherwise the sum of all possible

$$I(a_{i_0}; \cdots; a_{j_1}) I(a_{j_1}; \cdots; a_{j_2}) \cdots I(a_{j_m}; \cdots; a_{i_{n+1}})$$

Example 4.1.12.

$$\begin{aligned} I^{\sigma_1\sigma_0^2}(0; a_1, 0, 0, a_2, 0; 1) &= I(0; a_1) I(a_1; 0) I(0; 0) I(0; a_2, 0; 1) \\ &+ I(0; a_1) I(a_1; 0) I(0; 0, a_2; 0) I(0; 1) \\ &+ I(0; a_1) I(a_1; 0; 0) I(0; a_2; 0) I(0; 1) \end{aligned}$$

With this shorthand, (2.3.13) now reads

$$\Delta I(a_0; a_1, \cdots; a_n; a_{n+1}) = \sum_{0=i_0<i_1<\cdots<i_k<i_{k+1}=n+1} I(a_{i_0}; a_{i_1}, \cdots, a_{i_k}; a_{i_{k+1}}) \otimes I^{\sigma_{i_1}\cdots\sigma_{i_k}}(a_0; a_1, \cdots; a_n; a_{n+1}) \tag{4.1.7}$$

Since 0 could appear multiple times in the iterated integral, so the coproduct (4.1.7) may have collapsing terms and degenerates. For example, we just have

$$\Delta' I(0; a_1, 0, 0; 1) = I(0; a_1, 0; 1) \otimes (I(a_1; 0; 0) + I(0; 0; 1)) + I(0; a_1; 1) \otimes I(a_1; 0, 0; 1) \tag{4.1.8}$$

where

$$I(0;0,0;1)\otimes I(0;a_1;0),\quad I(0;0;1)\otimes I(0;a_1;0)I(0;0;1),\quad I(0;0;1)\otimes I(0;a_1,0;0)$$

are degenerates, and

$$I(0;a_1,0;1)\otimes I(a_1;0;0),\quad I(0;a_1,0;1)\otimes I(0;0;1)$$

simply collapsed into $I(0;a_1,0;1)\otimes(I(a_1;0;0)+I(0;0;1))$.

We are now ready to describe the variation matrix of

$$(-1)^d I(0;a_1,0^{n_1-1},\cdots,a_d,0^{n_d-1};1)$$

in $\mathbb{I}^{\mathrm{Symb}}(d)$. The inclusion of the sign $(-1)^d$ accounts for the sign in (2.3.3). Given the profinite nature of the variation matrix, this description provides a complete description of $V^{\mathbb{I}}$. We summarize this in the following Proposition.

Proposition 4.1.13. The first column of the variation matrix of

$$(-1)^d I(0;a_1,0^{n_1-1},\cdots,a_d,0^{n_d-1};1)$$

consist of entries of the form

$$(-1)^k I(0;a_{i_1},0^{m_{i_1}-1},\cdots,a_{i_k},0^{m_{i_k}-1};1)$$

where $(i_1,\cdots,i_k)$ is a subsequence of $(1,\cdots,d)$ and $m_{i_\alpha}\le n_{i_\alpha}$. They are ordered by the corresponding words $\sigma_{i_1}\sigma_0^{m_{i_1}-1}\cdots\sigma_{i_k}\sigma_0^{m_{i_k}-1}$, where we define $\sigma_{i_1}\sigma_0^{m_{i_1}-1}\cdots\sigma_{i_k}\sigma_0^{m_{i_k}-1}\prec\sigma_{j_1}\sigma_0^{p_{j_1}-1}\cdots\sigma_{j_l}\sigma_0^{p_{j_l}-1}$ if

- $m_{i_1}+\cdots+m_{i_k}<p_{j_1}+\cdots+p_{j_l}$

- else if $\sigma_{i_{k-r+1}}\sigma_0^{m_{i_{k-r+1}}-1}\cdots\sigma_{i_k}\sigma_0^{m_{i_k}-1} = \sigma_{j_{l-r+1}}\sigma_0^{p_{j_{l-r+1}}-1}\cdots\sigma_{j_l}\sigma_0^{m_{i_l}-1}$ and $i_{k-r} < j_{l-r}$

- else if $\sigma_0^{m_{i_{k-r}}-1}\sigma_{i_{k-r+1}}\cdots\sigma_{i_k}\sigma_0^{m_{i_k}-1} = \sigma_0^{p_{j_{l-r}}-1}\sigma_{j_{l-r+1}}\cdots\sigma_{j_l}\sigma_0^{m_{i_l}-1}$ and $m_{i_{k-r}} > p_{j_{l-r}}$

Note that this order is accordance with the order in Definition 3.1.10.

We then turn our attention to the remaining entries. It is not hard to see that the complementary entry of $(-1)^k I(0; a_{i_1}, 0^{m_{i_1}-1}, \cdots, a_{i_k}, 0^{m_{i_k}-1}; 1)$ with respect to $(-1)^l I(0; a_{j_1}, 0^{p_{j_1}-1}, \cdots, a_{j_l}, 0^{p_{j_l}-1}; 1)$ is

$$(-1)^{l-k} I^{\sigma_{i_1}\sigma_0^{m_{i_1}-1}\cdots\sigma_{i_k}\sigma_0^{m_{i_k}-1}}(0; a_{j_1}, 0^{p_{j_1}-1}, \cdots, a_{j_l}, 0^{p_{j_l}-1}; 1)$$

Where $(i_1, \cdots, i_k)$ is subsequence of $(j_1, \cdots, j_l)$, $(j_1, \cdots, j_l)$ is subsequence of $(1, \cdots, d)$, and $m_\alpha \le p_\alpha \le n_\alpha$.

Remark 4.1.14. The words $\sigma_{i_1}\sigma_0^{m_{i_1}-1}\cdots\sigma_{i_k}\sigma_0^{m_{i_k}-1}$ and $\sigma_{j_1}\sigma_0^{p_{j_1}-1}\cdots\sigma_{j_l}\sigma_0^{p_{j_l}-1}$ indicate the column and row index of

$$(-1)^{l-k} I^{\sigma_{i_1}\sigma_0^{m_{i_1}-1}\cdots\sigma_{i_k}\sigma_0^{m_{i_k}-1}}(0; a_{j_1}, 0^{p_{j_1}-1}, \cdots, a_{j_l}, 0^{p_{j_l}-1}; 1)$$

Since $m_\alpha < p_\alpha$, for all α, the variation matrix is evidently a lower-triangular unipotent matrix.

Example 4.1.15. We update Example 4.1.9. The variation matrix of $(-1)^2 I(0; a_1, 0, a_2; 1)$

is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -I(0;a_2;1) & 1 & 0 & 0 & 0 & 0 \\ -I(0;a_1;1) & 0 & 1 & 0 & 0 & 0 \\ I(0;a_1,a_2;1) & -I(0;a_1;a_2) & -I(a_1;a_2;1) & 1 & 0 & 0 \\ -I(0;a_1,0;1) & 0 & I(a_1;0;1) & 0 & 1 & 0 \\ I(0;a_1,0,a_2;1) & -I(0;a_1,0;a_2) & -I(a_1;0,a_2;1) & I(a_1;0;a_2) & -I(0;a_2;1) & 1 \end{bmatrix} \tag{4.1.9}$$

where

$$\begin{aligned} &-I(0;a_1;a_2) = (-1)I^{\sigma_2}(0;a_1,a_2;1), \quad -I(a_1;a_2;1) = (-1)I^{\sigma_1}(0;a_1,a_2;1), \\ &I(a_1;0;1) = I^{\sigma_1}(0;a_1,0;1), \quad -I(0;a_1,0;a_2) = (-1)I^{\sigma_2}(0;a_1,0,a_2;1), \\ &-I(a_1;0,a_2;1) = (-1)I^{\sigma_1}(0;a_1,0,a_2;1), \quad I(a_1;0;a_2) = I^{\sigma_1\sigma_2}(0;a_1,0,a_2;1), \\ &-I(0;a_2;1) = (-1)I^{\sigma_1\sigma_0}(0;a_1,0,a_2;1). \end{aligned}$$

4.1.3 Grading on variation matrix by weights

The entries of the variation matrix $V^{\mathbb{H}}$ have various weights, and we can divide them into different *weight blocks*. We start by dividing the rows according to weights in the first column. Following this, we apply the same divisions to the columns, ensuring that the weight of the (p,q)-th block is $p-q$. Notably, the (p,p)-th block is the identity matrix.

Example 4.1.16.

$$\left[\begin{array}{c|ccc|cc|c}
1 & 0 & & 0 & 0 & 0 & 0 \\
\hline
[x_2]_1 & 1 & & 0 & 0 & 0 & 0 \\
[x_1x_2]_1 & 0 & & 1 & 0 & 0 & 0 \\
\hline
[x_1,x_2]_{1,1} & [x_1]_1 & & [x_2]_1 - [x_1]_1 - [x_1]_0 & 1 & 0 & 0 \\
[x_1x_2]_2 & 0 & & [x_1x_2]_0 & 0 & 1 & 0 \\
\hline
[x_1,x_2]_{2,1} & [x_1]_2 & & -\frac{1}{2}[x_1]_0^2 + [x_1x_2]_0[x_2]_1 - [x_1]_2 - [x_2]_2 & [x_1]_0 & [x_2]_1 & 1
\end{array}\right]$$

Following Zhao's notation, we now define the τ matrix. The matrix $\tau(2\pi i)$ will play an important role in formulating Theorem 4.2.18 and 4.2.6.

Definition 4.1.17. $\tau_{n_1,\cdots,n_d}(a)$ is defined to be the diagonal square matrix of the same size and blocking as the variation matrix of $[x_1,\cdots,x_d]_{n_1,\cdots,n_d}$, where the (p,p)-th block is the a^p times the identity matrix.

Example 4.1.18.

$$\tau_{2,1}(2\pi i) = \left[\begin{array}{c|cc|cc|c}
1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & (2\pi i) & 0 & 0 & 0 & 0 \\
0 & 0 & (2\pi i) & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & (2\pi i)^2 & 0 & 0 \\
0 & 0 & 0 & 0 & (2\pi i)^2 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & (2\pi i)^3
\end{array}\right]$$

For simplicity, given a matrix V, we shall use $V_{p,q}$ to refer to its (p,q)-th block, and $V_k = \sum_{p+q=k} V_{p,q}$ to denote the submatrix consisting of all weight blocks of weight k.

The following useful lemma is straightforward.

Lemma 4.1.19. Suppose M is a matrix of the same dimension as the variation matrix, then $(\tau(a)^{-1}M\tau(a))_{p,q} = (2\pi i)^{q-p}M_{p,q}$.

Recall the symbol map $\Delta_{1,\cdots,1}$ is the repeated coproduct. A direct corollary of Theorem 4.1.8 derives a matrix computation of symbols of multiple polylogarithms.

Corollary 4.1.20. Suppose $k_1 + \cdots + k_m = n$, we have

$$\Delta_{k_1,\cdots,k_m}(V^T) = V_{k_1}^T \otimes \cdots \otimes V_{k_m}^T$$

In particular, by the definition of the symbol map $\Delta_{1,\cdots,1}$, the symbol of the top right entry of V^T is equal to the top right entry of $\Delta_{1,\cdots,1}(V^T) = (V_1^T)^{\otimes n}$.

Example 4.1.21. We calculate $V_1^T \otimes V_1^T \otimes V_1^T$ for the variation matrix V of $[x_1, x_2]_{2,1}$

$$\begin{bmatrix} 0 & [x_2]_1 & [x_1x_2]_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & [x_1]_1 & 0 & 0 \\ 0 & 0 & 1 & -[x_1]_0 - [x_1]_1 + [x_2]_1 & [x_1x_2]_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & [x_1]_0 \\ 0 & 0 & 0 & 0 & 0 & [x_2]_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\otimes 3} \tag{4.1.10}$$

and the top right entry is the symbol of $[x_1, x_2]_{1,1}$.

$$\Delta_{1,1,1}([x_1, x_2]_{2,1}) = [x_2]_1 \otimes [x_1]_1 \otimes [x_1]_0 - [x_1x_2]_1 \otimes [x_1]_0 \otimes [x_1]_0 + [x_1x_2]_1 \otimes [x_1x_2]_0 \otimes [x_2]_1$$
$$- [x_1x_2]_1 \otimes [x_1]_1 \otimes [x_1]_0 + [x_1x_2]_1 \otimes [x_2]_1 \otimes [x_1]_0 \tag{4.1.11}$$

Note that this is the same as (2.3.21).

4.1.4 Difference between $V^{\overline{\mathbb{H}}}$ and $V^{\mathbb{H}}$

Recall that INV gets rid of powers of πi, while $\overline{\mathrm{INV}}$ corresponds to the full Goncharov inversion, it is natural to ask how much do $\overline{\mathrm{INV}}(V^{\overline{\mathbb{H}}})$ and $V^{\mathbb{H}} = \mathrm{INV}(V^{\overline{\mathbb{H}}})$ differ. We now demonstrate that they differ only by a constant matrix multiplication. To establish this, we first need to prove a simple proposition.

Proposition 4.1.22. Suppose V is a variation matrix and denote $\omega = dV_1$. We then have $dV = \omega V$.

Proof. According to Corollary 4.1.20, $\Delta_{n-1,1}V_n^T = V_{n-1}^T \otimes V_1^T$. By Lemma 4.1.23, we know that $dV_n^T = V_{n-1}^T(dV_1^T)$, i.e. $dV_n = \omega V_{n-1}$. If we tally up all weights, we have $dV = \omega V$. □

Lemma 4.1.23. The $d = \varphi \circ \Delta_{n-1,1}$ on $\mathbb{H}_n^{\mathrm{Symb}}$, where φ takes $x \otimes y$ to xdy.

Proof. It is easy to show that $\varphi \circ \Delta_{n-1,1}[y_1, \dots, y_n | t_1, \dots, t_n]$ equals the right hand side of (3.1.17). We just need to check if it still holds for products, let P_1 and P_2 be symbols in weight k and l, respectively. We then have (for $n = k + l$)

$$\begin{aligned}\varphi \circ \Delta_{n-1,1}(P_1P_2) &= \varphi\big(\Delta_{k-1,1}(P_1)(P_2 \otimes 1) + (P_1 \otimes 1)\Delta_{l-1,1}(P_2)\big)\\ &= P_2\, dP_1 + P_1\, dP_2 = d(P_1P_2).\end{aligned} \tag{4.1.12}$$

The result follows. □

Theorem 4.1.24. $\overline{\mathrm{INV}}(V^{\overline{\mathbb{H}}}) = V^{\mathbb{H}}C$ for some constant matrix C with entries in $\mathbb{Q}[\pi i]$.

Proof. Since differential gets rid of constants, by Proposition 4.1.22 we have

$$d\overline{\mathrm{INV}}(V^{\overline{\mathbb{H}}}) = d\,\mathrm{INV}(V^{\overline{\mathbb{H}}}) = dV^{\mathbb{H}} = \omega V^{\mathbb{H}}$$

For convenience we denote $\overline{\mathrm{INV}}(V^{\overline{\mathbb{H}}}), V^{\mathbb{H}}$ as V_1, V_2 respectively. We then have

$$d(V_2^{-1}V_1) = -V_2^{-1}(dV_2)V_2^{-1}V_1 + V_2^{-1}(dV_1) = -V_2^{-1}\omega V_2 V_2^{-1}V_1 + V_2^{-1}\omega V_1 = 0$$

Thanks to Lemma 3.1.13 we know $V_2^{-1}V_1$ equals to some constant matrix C with entries in $\mathbb{Q}[\pi i]$.

We can evaluate C by simply replacing $[\mathbf{x}]_{\mathbf{n}}$ with 0 on both sides of the equation $\overline{\mathrm{INV}}(V^{\overline{\mathbb{H}}}) = V^{\mathbb{H}}C$. This corresponds to evaluating the regularized values of $\mathrm{Li}_{\mathbf{n}}(\mathbf{x})$, $\log(x)$ at the origin. As a result

$$\overline{\mathrm{INV}}(V^{\overline{\mathbb{H}}})\Big|_{[\mathbf{x}]_{\mathbf{n}}\to 0} = V^{\mathbb{H}}\Big|_{[\mathbf{x}]_{\mathbf{n}}\to 0} C = IC = C$$

□

We can actually do better with a matrix $\widetilde{C}$ that express as this difference totally rationally.

Corollary 4.1.25. $\overline{\mathrm{INV}}(V^{\overline{\mathbb{H}}})\tau(2\pi i) = V^{\mathbb{H}}\tau(2\pi i)\widetilde{C}$ for some constant matrix $\widetilde{C}$ with entries in $\mathbb{Q}$.

Proof. We take $\widetilde{C} = \tau(2\pi i)^{-1}C\tau(2\pi i)$, and under Theorem 4.1.24 we have

$$\overline{\mathrm{INV}}(V^{\overline{\mathbb{H}}})\tau(2\pi i) = V^{\mathbb{H}}C\tau(2\pi i) = V^{\mathbb{H}}\tau(2\pi i)\widetilde{C}$$

We only need to justify that $\widetilde{C}$ is rational. It is not hard to see that Theorem 2.3.7 tells us that the (p, q)-th block $C_{p,q}$ is of weight $p - q$, so it equals to $(2\pi i)^{p-q}$ times some rational matrix. On the other hand, according to Lemma 4.1.19, we know that $\widetilde{C}_{p,q}$ is equal to $(2\pi i)^{q-p}C_{p,q}$, which would be rational, and therefore so is $\widetilde{C}$. $\square$

Example 4.1.26.

$$\overline{\mathrm{INV}}\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ [x_2]_1 & 1 & 0 & 0 \\ [x_1x_2]_1 & 0 & 1 & 0 \\ [x_1,x_2]_{1,1} & [x_1]_1-[x_1^{-1}]_1+[x_2]_1 & 0 & 1 \end{bmatrix}\right)$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ [x_2]_1 & 1 & 0 & 0 \\ [x_1x_2]_1 & 0 & 1 & 0 \\ [x_1,x_2]_{1,1} & [x_1]_1 & \pi i-[x_1]_0-[x_1]_1+[x_2]_1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ [x_2]_1 & 1 & 0 & 0 \\ [x_1x_2]_1 & 0 & 1 & 0 \\ [x_1,x_2]_{1,1} & [x_1]_1 & -[x_1]_0-[x_1]_1+[x_2]_1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \pi i & 1 \end{bmatrix} \tag{4.1.13}$$

and

$$\widetilde{C} = \tau_{1,1}(2\pi i)^{-1}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \pi i & 1 \end{bmatrix}\tau_{1,1}(2\pi i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

4.2 Variation of mixed Hodge structures encoded by multiple polylogarithms

Zhao extended Goncharov's idea by treating iterated integrals as actual functions on $S_d(\mathbb{C})$. He defined a variation of mixed Hodge structures on $S_d(\mathbb{C})$ using filtrations on the matrix columns. Since these functions are multi-valued, he had to demonstrate that their monodromies are rational. He gave explicit formulas for the monodromy for multiple logarithms (i.e. $\mathrm{Li}_{1,\cdots,1}$). We will deal with the general case in Chapter 5.

In this section, we first review Zhao's contruction. Then we demonstrate that $V^{\mathbb{H}}$ under the one-form map gives the connection form of a flat connection on $\widehat{S}_d(\mathbb{C})$. Finally, we will prove that the flat sections of this connection form a lifted variation matrix $\widehat{V}$ which encodes a variation of mixed Hodge structures over $\widehat{S}_d(\mathbb{C})$. Moreover, the entries of first column of $\widehat{V}$ define lifted multiple polylogarithms $\widehat{\mathcal{L}}_{n_1,\cdots,n_d}$.

4.2.1 Realization of variation matrix

We write $H(d)$ for the elements of depth d (see Definition 3.3.6) in a free contraction Hopf algebra H.

We can "realize" the entries of these variation matrices as actual multi-valued multiple polylogarithmic functions. We call this process the realization map.

Definition 4.2.1. The realization map $\Re_{\mathbb{I}} : \mathbb{I}^{\mathrm{Symb}}(d) \to \mathcal{O}(\widetilde{S}_d(\mathbb{C}))$ is defined via Proposition 2.1.13 under the choice of $a_i = (x_i \cdots x_d)^{-1}$ with $0 < x_i < 1$, so that $1 <$

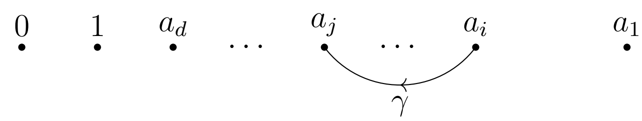


Figure 4.2.1: Realization map $\Re^{\mathbb{I}}$

$a_d < \cdots < a_1$. And we choose the canonical path γ such that $\gamma((0,1)) \subseteq \{\operatorname{Im} z < 0\}$ (See Figure 4.2.1).

$\Re_{\mathbb{I}}$ annihilates degenerate elements and $\Re_{\mathbb{I}}(I(0; a_1, 0^{n_1-1}, \cdots, a_d, 0^{n_d-1}; 1)) = \mathrm{Li}_{n_1,\cdots,n_d}(x_1, \cdots, x_d)$ as expected.

Definition 4.2.2. Similarly, realization map $\Re_{\overline{\mathbb{H}}} : \overline{\mathbb{H}}(d) \to \mathcal{O}(\tilde{S}_d(\mathbb{C}))$ can be defined by

$$\Re_{\overline{\mathbb{H}}}([x_1, \cdots, x_d]_{n_1,\cdots,x_d}) = \mathrm{Li}_{n_1,\cdots,n_d}(x_1, \cdots, x_d) \tag{4.2.1}$$

and

$$\Re_{\overline{\mathbb{H}}}([x_d^{-1}, \cdots, x_1^{-1}]_{n_d,\cdots,x_1}) = \mathrm{Li}_{n_d,\cdots,n_1}(x_d^{-1}, \cdots, x_1^{-1}) \tag{4.2.2}$$

which can simplified using Theorem 2.3.7. We could also define $\Re_{\mathbb{H}} : \mathbb{H}(d) \to \mathcal{O}(\tilde{S}_d(\mathbb{C}))$ to be the restriction $\Re_{\overline{\mathbb{H}}}|_{\mathbb{H}}$.

$\Re^{\mathbb{I}}$, $\Re^{\overline{\mathbb{H}}}$, $\Re^{\mathbb{H}}$ simply regard symbols as actual multi-valued functions, the difference is that they have different domains. Recall that Φ takes an iterated integral in $\mathbb{I}^{\mathrm{Symb}}$ and turns it into multiple polylogarithms in $\overline{\mathbb{H}}$, we have a more precise relation between $\Re^{\mathbb{I}}$, $\Re^{\overline{\mathbb{H}}}$, $\Re^{\mathbb{H}}$.

Proposition 4.2.3. For fixed d, the following diagram commutes

$$\begin{tikzcd}
\mathbb{I}^{\mathrm{Symb}}(d) \arrow[r, "\Phi"] \arrow[dr, "\Re"'] & \overline{\mathbb{H}}^{\mathrm{Symb}}(d) \arrow[r, "\overline{\mathrm{INV}}"] \arrow[d, "\Re"] & \mathbb{H}^{\mathrm{Symb}}[\pi i](d) \arrow[dl, "\Re"] \\
& \mathcal{O}(\tilde{S}_d(\mathbb{C})) &
\end{tikzcd}$$

Example 4.2.4. Suppose $V^{\overline{\mathbb{H}}}$, $V^{\mathbb{H}}$ are truncated at $[x_1, x_2]_{2,1}$ as in Example 4.1.10, then $\Re_{\overline{\mathbb{H}}}(V^{\overline{\mathbb{H}}})$ would be

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \mathrm{Li}_1(x_2) & 1 & 0 & 0 & 0 & 0 \\ \mathrm{Li}_1(x_1x_2) & 0 & 1 & 0 & 0 & 0 \\ \mathrm{Li}_{1,1}(x_1,x_2) & \mathrm{Li}_1(x_1) & \mathrm{Li}_1(x_2) - \mathrm{Li}_1(x_1^{-1}) & 1 & 0 & 0 \\ \mathrm{Li}_2(x_1x_2) & 0 & \log(x_1) + \log(x_2) & 0 & 1 & 0 \\ \mathrm{Li}_{2,1}(x_1,x_2) & \mathrm{Li}_2(x_1) & \substack{\mathrm{Li}_2(x_1^{-1}) - \mathrm{Li}_2(x_2) + \\ \mathrm{Li}_1(x_2)(\log(x_1) + \log(x_2))} & [x_1]_0 & \mathrm{Li}_1(x_2) & 1 \end{bmatrix} \tag{4.2.3}$$

and $\Re_{\mathbb{H}}(V^{\mathbb{H}})$ would be

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \mathrm{Li}_1(x_2) & 1 & 0 & 0 & 0 & 0 \\ \mathrm{Li}_1(x_1x_2) & 0 & 1 & 0 & 0 & 0 \\ \mathrm{Li}_{1,1}(x_1,x_2) & \mathrm{Li}_1(x_1) & \mathrm{Li}_1(x_2) - \mathrm{Li}_1(x_1) - \log(x_1) & 1 & 0 & 0 \\ \mathrm{Li}_2(x_1x_2) & 0 & \log(x_1) + \log(x_2) & 0 & 1 & 0 \\ \mathrm{Li}_{2,1}(x_1,x_2) & \mathrm{Li}_2(x_1) & \substack{-\mathrm{Li}_2(x_1) - \mathrm{Li}_2(x_2) - \frac{1}{2}\log^2(x_1) + \\ \mathrm{Li}_1(x_2)(\log(x_1) + \log(x_2))} & \log(x_1) & \mathrm{Li}_1(x_2) & 1 \end{bmatrix} \tag{4.2.4}$$

4.2.2 Variation of mixed Hodge structures of lifted multiple polylogarithms

As it turns out, the one-forms that we described in Definition 3.2.3, can be interpreted as the differential one-forms inside a connection form of a flat connection of a vector bundle over $\widehat{S}_d(\mathbb{C})$ whose flat sections are defined to be the lifted multiple polylogarithms. Furthermore, we can define a variation of mixed Hodge structures on this vector bundle by defining filtrations on these lifted multiple polylogarithms. This is described in [9]. However, Zhao only states and proves it explicitly for multiple logarithms $\mathrm{Li}_{1,\cdots,1}$.

For simplicity, denote either $\Re(V^{\overline{\mathbb{H}}})$ or $\Re(V^{\mathbb{H}})$ truncated at $\mathrm{Li}_{n_1,\cdots,n_d}(x_1,\cdots,x_d)$ by V. Suppose V is a $N\times N$ matrix and let $\{\mu_p\}$ be integers such that the (p,q)-weight block is the submatrix with indices (i,j) satisfying $\mu_{p-1} < i \le \mu_p$, $\mu_{q-1} < j \le \mu_q$. Additionally, let $\omega = dV_1$ be the differential of V_1, where V_1 is the submatrix of V consisting of weight 1 blocks. Zhao's connection theorem says

Theorem 4.2.5. [9] $\nabla = d - \omega$ defines a flat connection on the trivial bundle $S_d(\mathbb{C}) \times \mathbb{C}^N \to S_d(\mathbb{C})$, and the columns of $V\tau(2\pi i)$ generate the global sections of the local system corresponding to ∇.

Proof. We simply need to prove that $\nabla \circ \nabla V\tau(2\pi i) = 0$. By Proposition 4.1.22, we have $dV = \omega \wedge V$, so

$$0 = d^2V = d(\omega \wedge V) = d\omega \wedge V - \omega \wedge (\omega \wedge V) = (d\omega - \omega \wedge \omega) \wedge V$$

Since V is invertible, $d\omega - \omega \wedge \omega = 0$, i.e. ω is a flat connection form, and ∇ is a flat connection as in Definition 2.4.3 with the columns generating the global sections of the local system corresponding to ∇. $\square$

Using this, Zhao also stated his variation theorem

Theorem 4.2.6. [9] The columns $\{C_j\}_{j=1}^N$ of $V\tau(2\pi i)$ define a variation of Hodge structures over $S_d(\mathbb{C})$ as follows: Let $\{e_i\}_{i=1}^N$ denote the standard basis of $\mathbb{C}^N$. The

Hodge filtration and weight filtration are given by

$$F^{-p} = \mathbb{C}\langle\{e_i\}_{i=1}^{\mu_p}\rangle, \quad W_{1-2m} = W_{-2m} = \mathbb{Q}\langle\{C_j\}_{j\geq\mu_m}\rangle. \tag{4.2.5}$$

Proof. The k-th graded weight piece gr_k^W is the (k,k)-th weight block of $V\tau(2\pi i)$, which is $(2\pi i)^k$ times the identity matrix, thus obviously a direct sum of Hodge structures. To ensure that the weight filtration is well-defined under analytic continuation amounts to showing that the monodromies preserve the weight filtration. Zhao [26] gives explicit formulas for the monodromy in the case $\mathbf{n} = (1, \ldots, 1)$. We tackle the general case in chapter 5. Finally, Griffith transversality follows directly from the fact that $dV = \omega V$, which implies $dC_i = \omega C_i \subseteq \mathbb{C}\langle\{e_j\}_{j=1}^{\mu_{p-1}}\rangle \otimes \Omega_X^1$ for any $\mu_{p-1} < i \leq \mu_p$. $\square$

Example 4.2.7. Consider $V^{\mathbb{H}}$ truncated at $[x_1, x_2]_{1,1}$, we have

$$V = \left[\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline \mathrm{Li}_1(x_2) & 1 & 0 & 0 \\ \mathrm{Li}_1(x_1x_2) & 0 & 1 & 0 \\ \hline \mathrm{Li}_{1,1}(x_1,x_2) & \mathrm{Li}_1(x_1) & f(x_1,x_2) & 1 \end{array}\right]$$

$$V\tau(2\pi i) = \left[\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline \mathrm{Li}_1(x_2) & (2\pi i) & 0 & 0 \\ \mathrm{Li}_1(x_1x_2) & 0 & (2\pi i) & 0 \\ \hline \mathrm{Li}_{1,1}(x_1,x_2) & (2\pi i)\,\mathrm{Li}_1(x_1) & (2\pi i)f(x_1,x_2) & (2\pi i)^2 \end{array}\right]$$

where $f(x_1, x_2)$ denotes $-\log(x_1) - \mathrm{Li}_1(x_1) + \mathrm{Li}_1(x_2)$. The connection form is given by

$$\omega = dV_1 = \left[\begin{array}{c|cc|c} 0 & 0 & 0 & 0 \\ \hline -v_2 & 0 & 0 & 0 \\ -v_{1,2} & 0 & 0 & 0 \\ \hline 0 & -v_1 & -u_1 + v_1 - v_2 & 0 \end{array}\right]$$

$V\tau(2\pi i)$ encodes a variation of mixed Hodge structures, with the Hodge filtration being

- $F^n = 0$ for $n >= 0$
- $F^{-1} = \mathbb{C}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}$
- $F^{-2} = \mathbb{C}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\}$
- $F^n = \mathbb{C}^4$, for $n \leq -3$

And the weight filtration being

- $W_m = 0$ for $m < -6$

- $W_{-5} = W_{-6} = \mathbb{Q}\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ (2\pi i)^2 \end{bmatrix}\right\}$

- $W_{-3} = W_{-4} = \mathbb{C}\left\{\begin{bmatrix} 0 \\ (2\pi i) \\ 0 \\ (2\pi i)\operatorname{Li}_1(x_1) \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ (2\pi i) \\ (2\pi i) f(x_1, x_2) \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ (2\pi i)^2 \end{bmatrix}\right\}$

- $W_n = \mathbb{Q}\left\{\begin{bmatrix} 1 \\ \operatorname{Li}_1(x_2) \\ \operatorname{Li}_1(x_1 x_2) \\ \operatorname{Li}_{1,1}(x_1, x_2) \end{bmatrix}, \left[\begin{bmatrix} 0 \\ (2\pi i) \\ 0 \\ (2\pi i)\operatorname{Li}_1(x_1) \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ (2\pi i) \\ (2\pi i) f(x_1, x_2) \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ (2\pi i)^2 \end{bmatrix}\right]\right\}$,
  for $n \geq -2$.

We have shown that $\nabla = d - \omega$ is a flat connection on the trivial vector bundle $S_d(\mathbb{C}) \times \mathbb{C}^N \to S_d(\mathbb{C})$, with entries of V being its flat sections. Therefore, its pullback $\pi^*\nabla = d - \pi^*\omega$, under the projection $\pi : \widehat{S}_d(\mathbb{C}) \to S_d(\mathbb{C})$, is also a flat connection on the pullback trivial vector bundle, with connection form $\Omega = \pi^* V_1$.

Recall that the symbols on entries of V^T are $(V_1^T)^{\otimes n}$ from Corollary 4.1.20. It is natural to ask for the one-forms of the entries in V. We answer this with the following lemma.

Lemma 4.2.8. [2, Proposition 4.9] We have

$$w(V_n) = \frac{1}{n!} \sum_{k+l=n-1} (-1)^k \binom{n-1}{k} \Omega^k \omega \Omega^l. \tag{4.2.6}$$

Proof. Since $\Delta(V^T) = V^T \otimes V^T$, it follows that $\Delta_{n-1,1} V_n^T = V_{n-1}^T \otimes \Omega^T$. Therefore,

$$\Delta_{1,\ldots,1}(V^T) = I + \Omega^T + \Omega^T \otimes \Omega^T + \Omega^T \otimes \Omega^T \otimes \Omega^T + \cdots, \tag{4.2.7}$$

from which it follows that

$$w(V_n^T) = \frac{(-1)^{n+1}}{n!} \sum_{i=1}^{n} (-1)^{i-1} \binom{n-1}{i-1} (\Omega^T)^{i-1} \omega^T (\Omega^T)^{n-i}. \tag{4.2.8}$$

Hence,

$$w(V_n) = \frac{(-1)^{n+1}}{n!} \sum_{i=1}^{n} (-1)^{i-1} \binom{n-1}{i-1} \Omega^{n-i} \omega \Omega^{i-1}. \tag{4.2.9}$$

The result follows after reindexing $k = n - i$. □

Interestingly, up to some constants, $w(V)$ as a connection form is also flat.

Theorem 4.2.9. [2, Corollary 4.17] Define $\widehat{\omega}_n = (n-1)w(V_n)$, $\widehat{\omega} = \sum_n \widehat{\omega}_n$. Then $-\widehat{\omega}$ is a flat connection form. More specifically, it is the connection form of conjugated connection of $\pi^*\nabla$ under the automorphism $s \mapsto e^{\Omega} s$.

Proof. According to Proposition 2.4.4, the following lemma proves that $-\widehat{\omega}$ is the conjugated flat connections. □

We prove this in the following lemma with a slightly more general assumption of Ω only being nilpotent.

Lemma 4.2.10. Suppose Ω is any nilpotent matrix and $\omega = d\Omega$, then

$$\widehat{\omega} = de^{-\Omega} e^{\Omega} + e^{-\Omega} \omega e^{\Omega} = \sum_{n \geq 1} \frac{n-1}{n!} \sum_{k+l=n-1} (-1)^k \binom{n-1}{k} \Omega^k \omega \Omega^l. \tag{4.2.10}$$

Proof. First we compute

$$\begin{aligned}
de^{-\Omega}e^{\Omega}+e^{-\Omega}\omega e^{\Omega} &= d\left(\sum_{j\geq 0}(-1)^j\frac{\Omega^j}{j!}\right)\left(\sum_{r\geq 0}\frac{\Omega^r}{r!}\right)+\left(\sum_{k\geq 0}(-1)^k\frac{\Omega^k}{k!}\right)\omega\left(\sum_{l\geq 0}\frac{\Omega^l}{l!}\right)\\
&= \left(\sum_{p,q\geq 0}\frac{(-1)^{p+q+1}}{(p+q+1)!}\Omega^p\omega\Omega^q\right)\left(\sum_{r\geq 0}\frac{\Omega^r}{r!}\right)+\left(\sum_{k\geq 0}(-1)^k\frac{\Omega^k}{k!}\right)\omega\left(\sum_{l\geq 0}\frac{\Omega^l}{l!}\right)\\
&= \sum_{n\geq 1}\left(\sum_{\substack{p+q+r=n-1\\ p,q,r\geq 0}}\frac{(-1)^{p+q+1}}{(p+q+1)!r!}\Omega^p\omega\Omega^{q+r}+\sum_{\substack{k+l=n-1\\ k,l\geq 0}}\frac{(-1)^k}{k!l!}\Omega^k\omega\Omega^l\right).
\end{aligned}\tag{4.2.11}$$

The interior sum simplifies to

$$\sum_{k+l=n-1}\Omega^k\omega\Omega^l\left(\sum_{q+r=l}\frac{(-1)^{k+q+1}}{(k+q+1)!r!}+\frac{(-1)^k}{k!l!}\right)=\frac{n-1}{n!}\sum_{k+l=n-1}(-1)^k\binom{n-1}{k}\Omega^k\omega\Omega^l,\tag{4.2.12}$$

□

Let's denote the this conjugated flat connection as $\widehat{\nabla}$, and refer to it as the "lifted" connection, according to Proposition 2.4.4, $\widehat{V}=e^{-\Omega}(\pi^*V)$ are the flat sections under $\widehat{\nabla}$. We refer to it as the "lifted" variation matrices, and we define the entries of the first column of $\widehat{V}$ to be the lifted multiple polylogarithms.

Definition 4.2.11. The lifted multiple polylogarithm $\widehat{\mathcal{L}}_{n_1,\cdots,n_d}$ is defined as the entry in the first column of $\widehat{V}$ that corresponds to the entry for $\mathrm{Li}_{n_1,\cdots,n_d}$ in the first column of V.

Example 4.2.12. The variation matrix $\widehat{V}$ for $\mathrm{Li}_{1,1}(x_1, x_2)$ is

$$\left[\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline \widehat{\mathcal{L}}_{1,1}(x_1, x_2) & 0 & 0 & 1 \end{array}\right]$$

and the connection form is

$$\left[\begin{array}{c|cc|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline w_{1,1}(x_1, x_2) & 0 & 0 & 0 \end{array}\right]$$

Note that $\widehat{\mathcal{L}}_1$ and $\widehat{\omega}_1$ are 0.

We summerize previous discussions as the lifted connection theorem for reference.

Theorem 4.2.13. $\widehat{\nabla} = d - \widehat{\omega}$ is a flat connection on $\widehat{S}_d(\mathbb{C}) \times \mathbb{C}^N \to \widehat{S}_d(\mathbb{C})$, and the columns of $\widehat{V}\tau(2\pi i)$ generate the global sections of the local system corresponding to $\widehat{\nabla}$.

Proof. This is simply Theorem 4.2.9. $\square$

Because the one-form map factors through projection map P (see Remark 3.2.2), it gets rid of products. Therefore the connection form can be obtained simply by replacing $\mathrm{Li}_{n_1,\cdots,n_d}$ with $(n-1)w_{n_1,\cdots,n_d}$ (Theorem 4.2.9), and then modulo products.

There is a nice description of the entries of the lifted variation matrix $\widehat{V}$ due to the author (not included in [2]).

Theorem 4.2.14. Entries of $\widehat{V}$ are obtained by simply replacing $\mathrm{Li}_{n_1,\cdots,n_d}$ with $\widehat{\mathcal{L}}_{n_1,\cdots,n_d}$ for $n_1+\cdots+n_d \geq 2$, and log, Li_1 with zeros.

Proof. If we represent the replacement by ϕ, then by Lemma 4.2.16 below, we see that $\widehat{V} = \phi(V)$, and $\widehat{V}_1 = 0$. Then we use the fact that ϕ is multiplicative and that $\widehat{\mathcal{L}}_{n_1,\cdots,n_d}$ is the defined the bottom left corner of the lifted variation matrix of $\mathrm{Li}_{n_1,\cdots,n_d}$, i.e. $\widehat{\mathcal{L}}_{n_1,\cdots,n_d} = \phi(\mathrm{Li}_{n_1,\cdots,n_d})$. $\square$

Example 4.2.15. Consider the variation matrix $V^{\mathbb{H}}$ and lifted variation matrix $\widehat{V}$ for $\mathrm{Li}_{2,2,1}(x_1,x_2,x_3)$. Since they are huge matrices, we will only look at the complementary entry of $\mathrm{Li}_1(x_2x_3)$ with respect to $\mathrm{Li}_{2,2,1}(x_1,x_2,x_3)$, which are respectively

$$-\mathrm{Li}_2(x_1)\,\mathrm{Li}_2(x_2) - \mathrm{Li}_2(x_1)\,\mathrm{Li}_2(x_3) - \frac{1}{2}\,\mathrm{Li}_2(x_1)\log^2(x_2) + \mathrm{Li}_1(x_3)\,\mathrm{Li}_2(x_1)(\log(x_2)+\log(x_3))$$

and

$$-\widehat{\mathcal{L}}_2(u_1,v_1)\widehat{\mathcal{L}}_2(u_2,v_2) - \widehat{\mathcal{L}}_2(u_1,v_1)\widehat{\mathcal{L}}_2(u_3,v_3)$$

Lemma 4.2.16. Suppose H is a graded connected Hopf algebra, consider

$$\phi_{n,k} : H_n \xrightarrow{\Delta_{1,\cdots,1,n-k}} H_1^{\otimes k} \otimes H_{n-k} \xrightarrow{m^{\circ k}} H_n$$

We define $\phi : H \to H$ as the direct sum of $\phi_n : H_n \to H_n$, $\phi = \sum_{k=0}^{n}(-1)^k \phi_{n,k}/k!$, note that $\phi_0 = \mathrm{id}$. We claim that ϕ is multiplicative.

Proof. Suppose $a \in H_n, b \in H_m$. Recall if $\mathbf{n} = (n_1,\ldots,n_d) \in \mathbb{Z}^d_{\geq 0}$ with $|\mathbf{n}| = n$, $\Delta_{n_1,\cdots,n_d}(a) = \sum_i a^{\mathbf{n}}_{i1} \otimes \cdots \otimes a^{\mathbf{n}}_{id}$. Denote $a^{\mathbf{n}} = \sum_i a^{\mathbf{n}}_{i1} \cdots a^{\mathbf{n}}_{id}$ so that $\phi_{n,k}(a) =$

$a^{(1,\ldots,1,n-k)}$. Now it is not hard to see that

$$\begin{aligned}\Delta_{r_1,\ldots,r_d}(ab) &= \sum_{\substack{\mathbf{n}+\mathbf{m}=\mathbf{r}\\ \mathbf{n},\mathbf{m}\in\mathbb{Z}^d_{\geq 0}}} \left(\sum_i a^{\mathbf{n}}_{i1}\otimes\cdots\otimes a^{\mathbf{n}}_{id}\right)\left(\sum_j b^{\mathbf{m}}_{j1}\otimes\cdots\otimes b^{\mathbf{m}}_{jd}\right)\\ &= \sum_{\substack{\mathbf{n}+\mathbf{m}=\mathbf{r}\\ \mathbf{n},\mathbf{m}\in\mathbb{Z}^d_{\geq 0}}} \sum_i\sum_j a^{\mathbf{n}}_{i1}b^{\mathbf{m}}_{j1}\otimes\cdots\otimes a^{\mathbf{n}}_{id}b^{\mathbf{m}}_{jd},\end{aligned}$$

and

$$m^{\circ d}\circ\Delta_{r_1,\ldots,r_d}(ab) = \sum_{\substack{\mathbf{n}+\mathbf{m}=\mathbf{r}\\ \mathbf{n},\mathbf{m}\in\mathbb{Z}^d_{\geq 0}}} \sum_i\sum_j a^{\mathbf{n}}_{i1}\cdots a^{\mathbf{n}}_{id}\cdot b^{\mathbf{m}}_{j1}\cdots b^{\mathbf{m}}_{jd} = \sum_{\substack{\mathbf{n}+\mathbf{m}=\mathbf{r}\\ \mathbf{n},\mathbf{m}\in\mathbb{Z}^d_{\geq 0}}} a^{\mathbf{n}}b^{\mathbf{m}}$$

In particular, we get

$$\phi_r(ab) = \sum_{p=0}^{r}\frac{(-1)^p}{p!}\phi_{r,p}(ab) = \sum_{p=0}^{r}\frac{(-1)^p}{p!}\sum_{\substack{k+l=p\\ k,l\geq 0}}\binom{p}{k}a^{(1,\ldots,1,n-k)}b^{(1,\ldots,1,m-l)} \tag{4.2.13}$$

On the other hand, we have

$$\begin{aligned}\phi_n(a)\phi_m(b) &= \left(\sum_{k=0}^{n}\frac{(-1)^k}{k!}\phi_{n,k}(a)\right)\left(\sum_{l=0}^{m}\frac{(-1)^l}{l!}\phi_{m,l}(b)\right)\\ &= \sum_{p=0}^{r}\sum_{\substack{k+l=p\\ k,l\geq 0}}\frac{(-1)^p}{k!l!}a^{(1,\ldots,1,n-k)}b^{(1,\ldots,1,m-l)}\end{aligned} \tag{4.2.14}$$

Both equations above are equivalent. $\square$

Remark 4.2.17. Note that $\phi_1 = 0$. This explains why $\mathrm{Li}_1, \log$ are replaced with zero in Theorem 4.2.14.

We are now able to generalize Zhao's variation theorem to a lifted variation theorem.

Theorem 4.2.18. ([2], Theorem 5.8) The columns $\{C_j\}_{j=1}^N$ of $\widehat{V}\tau(2\pi i)$ define a variation of Hodge structures over $\widehat{S}_d(\mathbb{C})$ with Hodge filtration and weight filtration given by

$$F^{-p} = \mathbb{C}\langle\{e_i\}_{i=1}^{\mu_p}\rangle,\quad W_{1-2m} = W_{-2m} = \mathbb{Q}\langle\{C_j\}_{j\geq\mu_m}\rangle \tag{4.2.15}$$

Proof. Griffith transversality follows from the fact that $d\widehat{V} = \widehat{\omega}\widehat{V}$. All else follows from Theorem 4.2.6. $\square$

4.3 Applications of variation matrix

4.3.1 Single-valued multiple polylogarithms

We first recall the definition of single-valued polylogarithms from [4].

Definition 4.3.1. The single-valued polylogarithm $\widehat{\mathcal{L}}_n(z)$ is defined to be

$$\mathrm{ReIm}_n \left(\sum_{r=0}^{n-1} \frac{2^r B_r}{r!} \mathrm{Li}_{n-r}(z) \log^r |z| \right) \tag{4.3.1}$$

Here ReIm is Re when n is odd and Im when n is even, B_r are Bernoulli numbers.

This notion can be further generalized using the variation matrix.

Definition 4.3.2. [9] The single-valued multiple polylogarithm $\mathcal{L}_{n_1,\cdots,n_d}$ is defined to be $\dfrac{i^{2\lfloor (n_1+\cdots+n_d)/2 \rfloor - 1}}{2}$ multiplies the bottom left entry of $\log\left(\tau(i)V\tau(-1)\overline{V}^{-1}\tau(i)\right)$, where $V = \Re(V^{\overline{\mathbb{H}}})$ or $V = \Re(V^{\mathbb{H}})$ is the variation matrix of $\mathrm{Li}_{n_1,\cdots,n_d}(x_1,\cdots,x_d)$.

Remark 4.3.3. To see it is single-valued, consider monodromy $\mathcal{M}V\tau(2\pi i) = V\tau(2\pi i)\widetilde{M}$, where $\widetilde{M}$ is a rational matrix, then

$$\begin{aligned}
&\mathcal{M}\log\left(\tau(i)V\tau(-1)\overline{V}^{-1}\tau(i)\right) \\
&\quad= \log\left(\tau(i)V\tau(2\pi i)\widetilde{M}\tau(2\pi i)^{-1}\tau(-1)\overline{V\tau(2\pi i)\widetilde{M}\tau(2\pi i)^{-1}}^{-1}\tau(i)\right) \\
&\quad= \log\left(\tau(i)V\tau(2\pi i)\widetilde{M}\tau(1/2\pi i)\tau(-1)\tau(-2\pi i)\widetilde{M}^{-1}\tau(-1/2\pi i)\overline{V}^{-1}\tau(i)\right) \\
&\quad= \log\left(\tau(i)V\tau(-1)\overline{V}^{-1}\tau(i)\right)
\end{aligned} \tag{4.3.2}$$

Note that $\tau(a)\tau(b) = \tau(ab)$, $\tau(a)^k = \tau(a^k)$ for $k \in \mathbb{Z}$ and $\overline{\tau(a)} = \tau(\overline{a})$.

Example 4.3.4. Consider the variation matrix of $\mathrm{Li}_{1,1}(x_1,x_2)$

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \mathrm{Li}_1(x_2) & 1 & 0 & 0 \\ \mathrm{Li}_1(x_1x_2) & 0 & 1 & 0 \\ \mathrm{Li}_{1,1}(x_1,x_2) & \mathrm{Li}_1(x_1) & -\log(x_1) - \mathrm{Li}_1(x_1) + \mathrm{Li}_1(x_2) & 1 \end{bmatrix}$$

we have

$$\log\left(\tau(i)V\tau(-1)\overline{V}^{-1}\tau(i)\right) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2i\mathcal{L}_1(x_2) & 0 & 0 & 0 \\ 2i\mathcal{L}_1(x_1x_2) & 0 & 0 & 0 \\ -2i\mathcal{L}_{1,1}(x_1,x_2) & 2i\mathcal{L}_1(x_1) & 2i(\log|x_1| - \mathcal{L}_1(x_1) + \mathcal{L}_1(x_2)) & 0 \end{bmatrix} \tag{4.3.3}$$

where $\mathcal{L}_1(x) = \mathrm{Re}\,\mathrm{Li}_1(x) = -\log|1-x|$ and

$$\mathcal{L}_{1,1}(x_1,x_2) = \mathrm{Im}\,\mathrm{Li}_{1,1}(x_1,x_2) + \mathrm{Im}\,\mathrm{Li}_1(x_1x_2)\,\mathrm{Re}\log(x_1)$$

$$-\mathrm{Im}\,\mathrm{Li}_1(x_2)\,\mathrm{Re}\,\mathrm{Li}_1(x_1) + \mathrm{Im}\,\mathrm{Li}_1(x_1x_2)\,\mathrm{Re}\,\mathrm{Li}_1(x_1) - \mathrm{Im}\,\mathrm{Li}_1(x_1x_2)\,\mathrm{Re}\,\mathrm{Li}_1(x_2) \tag{4.3.4}$$

Re, Im denotes the real and imaginary parts.

In [12] (Theorem 2.10, 4.7), Zickert proved that $\mathcal{L}_n \circ \pi$ coincides with the real or imaginary part of $\widehat{\mathcal{L}}_n$ on $\widehat{\mathcal{B}}_n(\mathbb{C})$.

Here we would like to give an alternative formulation of this result in terms of the variation matrices. Let us assume $\widehat{V}$ to be the variation matrix in depth 1.

$$\begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ \widehat{\mathcal{L}}_2(u,v) & 0 & 1 & & \\ \widehat{\mathcal{L}}_3(u,v) & 0 & 0 & 1 & \\ \widehat{\mathcal{L}}_4(u,v) & 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{4.3.5}$$

Immediately we notice $\widehat{V}_i \widehat{V}_j = 0$ for $i, j \geq 2$, so we have

$$\widehat{V}^{-1} = (I + \widehat{V}_1 + \widehat{V}_2 + \cdots)^{-1} = I - \widehat{V}_1 - \widehat{V}_2 - \cdots$$

$$\log(\widehat{V}) = \log(I + \widehat{V}_1 + \widehat{V}_2 + \cdots) = \widehat{V}_1 + \widehat{V}_2 + \cdots$$

Then we can make the following simplifications

$$\begin{aligned}
&\log\left(\tau(i) V \tau(-1) \overline{V}^{-1} \tau(i)\right) \\
&= \log\left(\tau(i) e^{\Omega} \widehat{V} \tau(-1) \overline{(e^{\Omega}\widehat{V})}^{-1} \tau(i)\right) \\
&= \log\left(\tau(i) e^{\Omega} \widehat{V} \tau(-1) \overline{\widehat{V}}^{-1} e^{-\overline{\Omega}} \tau(i)\right) \\
&= \log\left(e^{i\Omega}\left(I + \sum_{k\geq 2} i^k \widehat{V}_k\right)\left(I - \sum_{k\geq 2} (-i)^k \overline{\widehat{V}_k}\right) e^{i\overline{\Omega}}\right) \\
&= \log\left(e^{i\Omega}\left(I + \sum_{k\geq 2} i^k \left(\widehat{V}_k - (-1)^k \overline{\widehat{V}_k}\right)\right) e^{i\overline{\Omega}}\right)
\end{aligned} \tag{4.3.6}$$

The third equality is a bit tricky, but this can be verified by comparing with the (p, q)-th weight blocks.

For simplicity, we denote $W_k = \left(\widehat{V}_k - (-1)^k \overline{\widehat{V}_k}\right)$. Then $\exp\left(\sum_{k\geq 2} i^k W_k\right) = I + \sum_{k\geq 2} i^k W_k$. Apply Baker-Campbell-Hausdorff formula to the previous result

$$\log\left(e^{i\Omega} \exp\left(\sum_{k\geq 2} i^k W_k\right) e^{i\overline{\Omega}}\right)$$

which renders sums of folded Lie brackets of the format

$$[\cdots,[i^k W_k,[\cdots]],\cdots], \quad [\cdots]$$

where the omitted arguments are either Ω or $\overline{\Omega}$.

Example 4.3.5. We work out all the Lie brackets up to weight 4.

$$\begin{aligned}
&\log\left(e^{i\Omega}\exp\left(\sum_{k\geq 2} i^k W_k\right)e^{i\overline{\Omega}}\right) - I - \sum_{k\geq 2} i^k W_k \\
&= i\left(\Omega+\overline{\Omega}\right) + i^2\left(W_2 + \frac{1}{2}[\Omega,\overline{\Omega}]\right) \\
&+ i^3\left(W_3 + \frac{1}{2}[\Omega-\overline{\Omega},W_2] + \frac{1}{12}[\Omega-\overline{\Omega},[\Omega,\overline{\Omega}]]\right) \\
&+ i^4\left(W_4 + \frac{1}{2}[\Omega-\overline{\Omega},W_3] + \frac{1}{8}[\Omega-\overline{\Omega},[\Omega-\overline{\Omega},W_2]] - \frac{1}{24}[\Omega+\overline{\Omega},[\Omega+\overline{\Omega},W_2]]\right. \\
&\qquad\left. -\frac{1}{96}\left([\Omega+\overline{\Omega},[\Omega+\overline{\Omega},[\Omega,\overline{\Omega}]]] - [\Omega-\overline{\Omega},[\Omega-\overline{\Omega},[\Omega,\overline{\Omega}]]]\right)\right) \\
&+\cdots
\end{aligned} \tag{4.3.7}$$

If we restrict our attention to the bottom left entry, we will show that (4.3.6) establishes an equality between the left hand side $\mathcal{L}\circ\pi(u,v)$ and the right hand side which is the real or imaginary part of $\widehat{\mathcal{L}}(u,v)$ plus a bunch of folded Lie brackets (see Example 4.3.5).

Imitating Zickert [12], let us show that these folded Lie brackets can be realized as a composition of maps that always starts with $\widehat{\delta}_n : \widehat{\mathcal{B}}_n(\mathbb{C}) \to \widehat{\mathcal{B}}_{n-1}(\mathbb{C})\otimes\mathbb{C}$ for $n>2$ or $\widehat{\delta}_2 : \widehat{\mathcal{B}}_2(\mathbb{C}) \to \bigwedge^2\mathbb{C}$. First we define $\Psi_s, s<n-1$ similar to [12] as $(\delta_{n-s+1}\otimes 1\otimes\cdots\otimes 1)\cdots(\widehat{\delta}_{n-1}\otimes 1)\circ\widehat{\delta}_n$

$$\begin{aligned}
&\Psi_s : \mathbb{Z}[\widehat{\mathbb{C}}] \to \widehat{\mathcal{B}}_{n-s}(\widehat{\mathbb{C}})\otimes\mathbb{C}^{\otimes s}, \quad [(u,v)]\mapsto[(u,v)]\otimes u^{\otimes s}, \quad s<n-2 \\
&\Psi_{n-2} : \mathbb{Z}[\widehat{\mathbb{C}}] \to \wedge^2\mathbb{C}\otimes\mathbb{C}^{\otimes(n-2)}, \quad (u\wedge v)\otimes u^{\otimes s}
\end{aligned} \tag{4.3.8}$$

So $\Psi_s(\widehat{V}^T) = \widehat{V}^T \otimes \Omega^{T^{\otimes s}}, s < n-2,\ \Psi_{n-2}(\widehat{V}^T) = (\Omega^T \wedge \Omega^T) \otimes \Omega^{T^{\otimes s}}$.

Ψ_s always starts with $\widehat{\delta}_n$, then any of these Lie brackets can start some Ψ_s. For example, $[\Omega, [\overline{\Omega}, [i^3 W_3, [\Omega, \overline{\Omega}]]]]$ can be realized as the composition of maps

$$\begin{tikzcd}
\widehat{V}_7^T \arrow[rr, "\Psi_4"] & & \widehat{V}_3^T \otimes \Omega^{T^{\otimes 3}} \arrow[d, "\cdot^T \circ (i^k(\mathrm{id} - (-1)^k \bar{\cdot}) \otimes \mathrm{id} \otimes \mathrm{id} \otimes \bar{\cdot} \otimes \bar{\cdot})"] & \\
& & \overline{\Omega} \otimes \overline{\Omega} \otimes \Omega \otimes \Omega \otimes i^3 W_3 \arrow[r] & {[\Omega, [\overline{\Omega}, [W_3, [\Omega, \overline{\Omega}]]]]}
\end{tikzcd}$$

Here $\bar{\cdot}$ stands for conjugation, $\cdot^T$ stands for transpose, and the last map is a sum of products of $i^3 W_3, \Omega, \overline{\Omega}$. Similarly $[\Omega, \overline{\Omega}]$ can be realized as

$$\widehat{V}_4^T \xrightarrow{\cdot^T \circ \Psi_3} \Omega^{\otimes 2} \otimes (\Omega \wedge \Omega) \longrightarrow [\Omega, [\Omega, [\Omega, \overline{\Omega}]]]$$

The last map is a sum of products of $\Omega, \overline{\Omega}$.

Now that if we decompose $\Omega, \overline{\Omega}$ into $\operatorname{Re}\Omega \pm i \operatorname{Im}\Omega$. Then the coefficients obtained by applying Baker-Campbell-Hausdorff formula should correspond to the coefficients in Theorem 2.10 in [12].

4.3.2 Recursion of one-forms

In [15], Greenberg found a recursion formula of the one-forms. We say a one-form is of weight n if it can be written as $\sum_i p_i du_i + \sum_{j,k} q_{j,k} dv_{j,k}$, where $p_i, q_{j,k}$ are polynomials in u, v variables of degree $n-1$. We use $\left(\Omega_{[u_i, v_{j,k}]/\mathbb{Q}}\right)_n$ to denote weight n one-forms.

Example 4.3.6.

$$\left(\Omega_{[u_i, v_{j,k}]/\mathbb{Q}}\right)_1 = \bigoplus_{i,j,k} \mathbb{Q} du_i + \mathbb{Q} dv_{j,k} \tag{4.3.9}$$

$$\left(\Omega_{[u_i, v_{j,k}]/\mathbb{Q}}\right)_2 = \oplus_{i,j,k} (\mathbb{Q} u_j + \mathbb{Q} v_{j,k}) du_i \oplus \oplus_{i,j,k,l} (\mathbb{Q} u_i + \mathbb{Q} v_{i,j}) dv_{k,l} \tag{4.3.10}$$

We provide a reformulation of the recursion formula in terms of the variation matrix.

Theorem 4.3.7. Suppose $\Omega = V_1^{\mathbb{H}}$, $\omega = d\Omega$, Then the bottom left entry of the following direct identity is the recursion formula

$$[n!w(V_n), \Omega] = (n+1)!w(v_{n+1}) \tag{4.3.11}$$

Note that the first argument on the left hand side Lie bracket and the right hand side is $n!w(V_n)$, $(n+1)!w(V_{n_1})$ respectively according to (4.2.8).

Proof. According to (4.2.8), (4.3.11) is really the following direct computation.

$$\left[\sum_{k+l=n-1} (-1)^{k+1}\binom{n-1}{k}\Omega^k\omega\Omega^l, \Omega\right] = \sum_{k+l=n} (-1)^{k+1}\binom{n}{k}\Omega^k\omega\Omega^l \tag{4.3.12}$$

□

Example 4.3.8. Take the variation matrix for $\mathrm{Li}_{2,1}(x_1, x_2)$ for an example, we have

$$w(V) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -v_2 & 0 & 0 & 0 & 0 & 0 \\ -v_{1,2} & 0 & 0 & 0 & 0 & 0 \\ w_{1,1}(x_1, x_2) & -v_1 & v_1 - v_2 - u_1 & 0 & 0 & 0 \\ w_2(x_1 x_2) & 0 & u_1 + u_2 & 0 & 0 & 0 \\ w_{2,1}(x_1, x_2) & w_2(x_1) & -w_2(x_1) - w_2(x_2) & u_1 & -v_2 & 0 \end{bmatrix}$$

$$\Omega = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -v_2 & 0 & 0 & 0 & 0 & 0 \\ -v_{1,2} & 0 & 0 & 0 & 0 & 0 \\ 0 & -v_1 & v_1 - v_2 - u_1 & 0 & 0 & 0 \\ 0 & 0 & u_1 + u_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_1 & -v_2 & 0 \end{bmatrix}$$

So the bottom left entry of $[2w(V_2), \Omega] = 6w(V_3)$ gives

$$w_{2,1}(x_1, x_2) = \frac{1}{3}\left(-v_2 w_2(x_1) + v_{1,2}(w_2(x_1) + w_2(x_2)) - u_1 w_{1,1}(x_1, x_2) + v_2 w_2(x_1 x_2)\right) \tag{4.3.13}$$

Chapter 5: Monodromy of variation matrices

Zhao in [9] gave formulas for the monodromy of multiple logarithms $\mathrm{Li}_{1,\cdots,1}$, by interpreting them as iterated integrals over $S_d(\mathbb{C})$ (see Section 2.3.5). He claimed that the monodromy of a general multiple polylogarithm can be computed by taking the limit of variation mixed Hodge structures of multiple logarithms, but gave neither an explicit formula nor an algorithm.

In this chapter, we present a fresh viewpoint by interpreting multiple polylogarithms as iterated integrals on $\mathbb{P}^1 - \{0, 1, \infty\}$. Using Proposition 2.1.13, we show that when a multiple polylogarithm undergoes a monodromy, it corresponds to a continuous deformation of the integration path, providing an alternative geometric interpretation. This perspective naturally give rise to explicit formulas, which are also implemented by the author using Mathematica.

5.1 Deformation of integration paths of iterated integrals under monodromy

Recall from Section 2.3.5 that ν_i, $\nu_{j,k}$ are loops around divisors $x_i = 0$ and $x_j \cdots x_k = 1$ in $S_d(\mathbb{C})$, and that $\mathcal{M}_{\nu_i}$, $\mathcal{M}_{\nu_{j,k}}$ denote the monodromy operators on iterated integrals if we think of them as multi-valued functions on $S_d(\mathbb{C})$.

Recall the realization map $\Re_{\mathbb{I}} : \mathbb{I}^{\text{Symb}} \to \mathcal{O}(\widetilde{S}_d(\mathbb{C}))$ from Definition 4.2.1. Without loss of generality, we fix some choice of $a_i = (x_i \cdots x_d)^{-1}$ with $0 < x_i < 1$, and paths γ from a_i to a_j such that the image of γ is below the real axis $\{\operatorname{Im} z = 0\}$. Since monodromies of $I(a_{i_0}; a_{i_1}, \cdots, a_{i_n}; a_{i_{n+1}})$ can be converted into deformations of the integration path of $I_\gamma(a_{i_0}; a_{i_1}, \cdots, a_{i_n}; a_{i_{n+1}})$, as discussed in Remark 2.1.12 and Remark 2.1.22, we are able to give a complete formulation of monodromies of realizations of iterated integrals in $\mathbb{I}^{\text{Symb}}(d)$.

5.1.1 Deformation of integration path under $\mathcal{M}_{\nu_{i_0}}$

If we fix a path from a to b, then the homotopy classes of paths in $\mathbb{C} - \{a_i\}$ from a to b can be identified with $\pi_1(\mathbb{C} - \{a_i\})$, so we could write any path from a to b as a loop in $\pi_1(\mathbb{C} - \{a_i\})$.

Theorem 5.1.1.

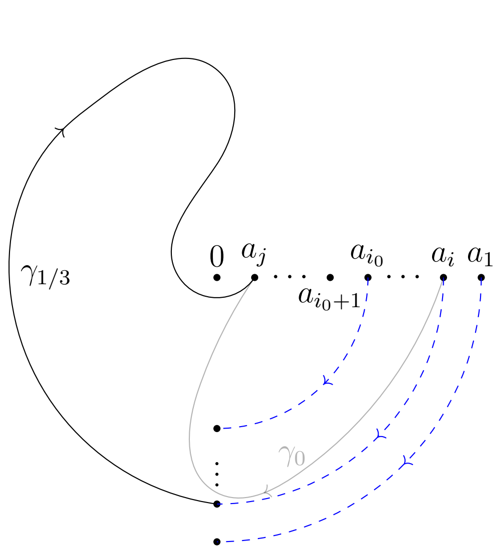
i. If $1 \le i \le i_0 < j \le d+1$, then

$$\mathcal{M}_{\nu_{i_0}} I(a_i; \cdots; a_j) = I_{\sigma_{i_0+1} \cdots \sigma_{j-1} \sigma_0}(a_i; \cdots; a_j) \tag{5.1.1}$$

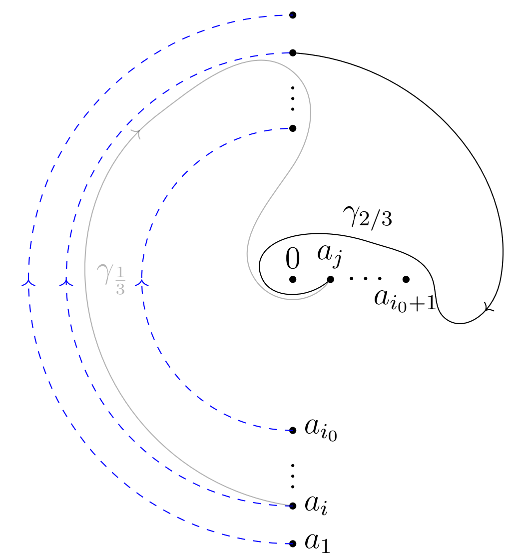
ii. If $1 \le i \le i_0$, then

$$\mathcal{M}_{\nu_{i_0}} I(0; \cdots; a_i) = I_{\sigma_0^{-1} \sigma_d^{-1} \cdots \sigma_{i_0+1}^{-1}}(0; \cdots; a_i) \tag{5.1.2}$$

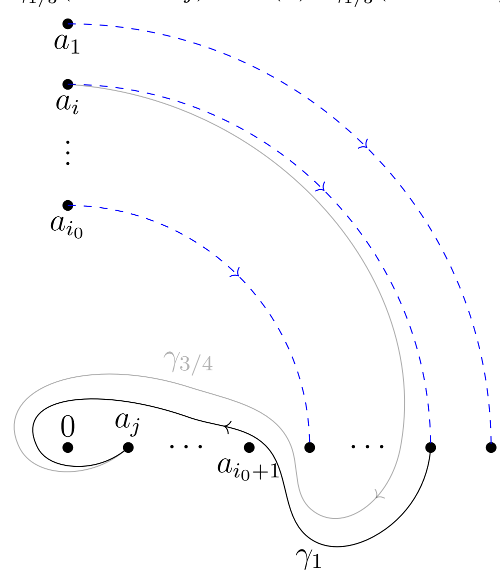
Proof. We choose ν_{i_0} to be a very small counterclockwise circle around 0, causing $a_1, \cdots, a_{i_0}$ to move around 0 in clockwise, concentric circles.



(a) $I_{\gamma_0}(a_0; \cdots; a_j)$ to $I_{\gamma_{1/3}}(a_0; \cdots; a_j)$



(b) $I_{\gamma_{1/3}}(a_0; \cdots; a_j)$ to $I_{\gamma_{2/3}}(a_0; \cdots; a_j)$



(c) $I_{\gamma_{2/3}}(a_0; \cdots; a_j)$ to $I_{\gamma_1}(a_0; \cdots; a_j)$

Figure 5.1.1: Deformation of $I(a_0; \cdots; a_j)$

To justify (5.1.1), first we deform γ_0 to $\gamma_{1/3}$ as $a_1, \cdots, a_{i_0}$ moves clockwise by $\pi/2$. This is shown in Figure 5.1.1a, where the faint path is γ_0, while the dashed paths are traces of $a_1, \cdots, a_{i_0}$. Then we deform $\gamma_{1/3}$ to $\gamma_{2/3}$ as $a_1, \cdots, a_{i_0}$ move clockwise by π. This is illustrated in Figure 5.1.1b. Lastly we deform $\gamma_{2/3}$ to γ_1 as $a_1, \cdots, a_{i_0}$ move clockwise by $\pi/2$, and back to where they started. This is presented in Figure 5.1.1c.

(5.1.2) is similar, and the deformation from γ_0 to γ_1 is shown in Figure 5.1.2.

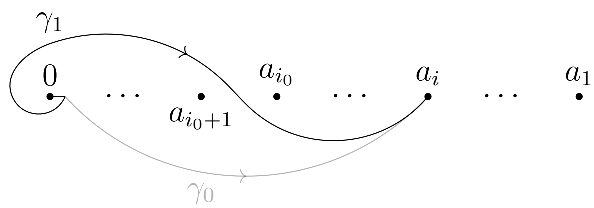


Figure 5.1.2: Deformation from $I_{\gamma_0}(0; \cdots; a_i)$ to $I_{\gamma_1}(0; \cdots; a_i)$

$\square$

5.1.2 Deformation of integration path under $\mathcal{M}_{\nu_{i_0},\nu_{j_0}}$

Theorem 5.1.2.

i.

$$\begin{aligned}
&\mathcal{M}_{\nu_{i_0,j_0}} I(a_{i_0}; \cdots; a_{j_0+1}) \\
&\qquad = I_{\sigma_{i_0+1}\cdots\sigma_{j_0}\sigma_{j_0+1}^{-1}\cdots\sigma_{i_0}^{-1}\sigma_{j_0}^{-1}\cdots\sigma_{i_0+1}^{-1}\sigma_{i_0}\cdots\sigma_{j_0}}(a_{i_0}; \cdots; a_{j_0+1}) \\
&\qquad = I_{\sigma_{i_0+1}\cdots\sigma_{j_0}\sigma_{j_0}^{-1}\cdots\sigma_{i_0+1}^{-1}\sigma_{j_0}^{-1}\cdots\sigma_{i_0+1}^{-1}\sigma_{i_0+1}\cdots\sigma_{j_0}}(a_{i_0}; \cdots; a_{j_0+1}) \\
&\qquad = I(a_{i_0}; \cdots; a_{j_0+1})
\end{aligned} \tag{5.1.3}$$

ii. If $j_0 + 1 < j \leq d + 1$, then

$$\begin{aligned}
\mathcal{M}_{\nu_{i_0,j_0}} I(a_{i_0}; \cdots; a_j) &= I_{\sigma_{i_0+1}\cdots\sigma_{j_0}\sigma_{j_0+1}^{-1}\cdots\sigma_{i_0}^{-1}}(a_{i_0}; \cdots; a_j) \\
&= I_{\sigma_{i_0+1}\cdots\sigma_{j_0}\sigma_{j_0+1}^{-1}\cdots\sigma_{i_0+1}^{-1}}(a_{i_0}; \cdots; a_j)
\end{aligned} \tag{5.1.4}$$

iii. If $1 \leq i < i_0$, then

$$\mathcal{M}_{\nu_{i_0,j_0}} I(a_i; \cdots ; a_{j_0+1}) = I_{\sigma_{j_0}^{-1} \cdots \sigma_{i_0+1}^{-1} \sigma_{i_0} \cdots \sigma_{j_0}}(a_i; \cdots ; a_{j_0+1}) \tag{5.1.5}$$

iv.

$$\begin{aligned} \mathcal{M}_{\nu_{i_0,j_0}} I(0; \cdots ; a_{i_0}) &= I_{\sigma_{i_0} \cdots \sigma_{j_0} \sigma_{j_0+1}^{-1} \cdots \sigma_{i_0+1}^{-1}}(0; \cdots ; a_{i_0}) \\ &= I_{\sigma_{i_0+1} \cdots \sigma_{j_0} \sigma_{j_0+1}^{-1} \cdots \sigma_{i_0+1}^{-1}}(0; \cdots ; a_{i_0}) \end{aligned} \tag{5.1.6}$$

v.

$$\mathcal{M}_{\nu_{i_0,j_0}} I(0; \cdots ; a_{j_0+1}) = I_{\sigma_{j_0}^{-1} \cdots \sigma_{i_0+1}^{-1} \sigma_{i_0} \cdots \sigma_{j_0}}(0; \cdots ; a_{j_0+1}) \tag{5.1.7}$$

Proof. We choose $\nu_{i_0 j_0}$ to be the loop in $S_d(\mathbb{C})$ where x_{i_0} traces through the path $\alpha \varepsilon \alpha^{-1}$ (see Figure 5.1.3) and $\{x_j\}_{j \neq i_0}$ stay still. Here α is an arc and ε is a sufficiently small circle around $(x_{i_0+1} \cdots x_{j_0})^{-1}$.

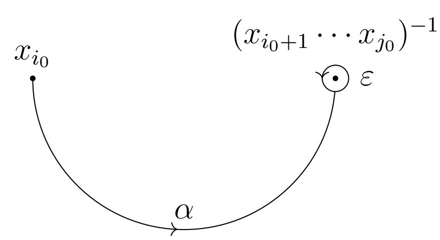
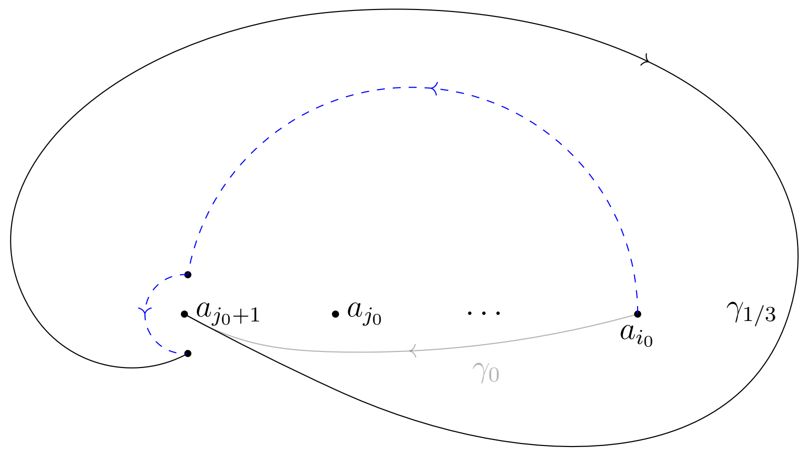
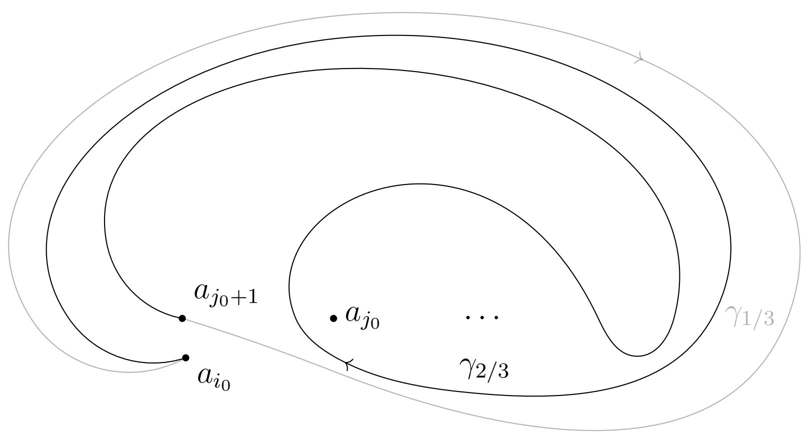


Figure 5.1.3: Choice of ν_{i_0,j_0}

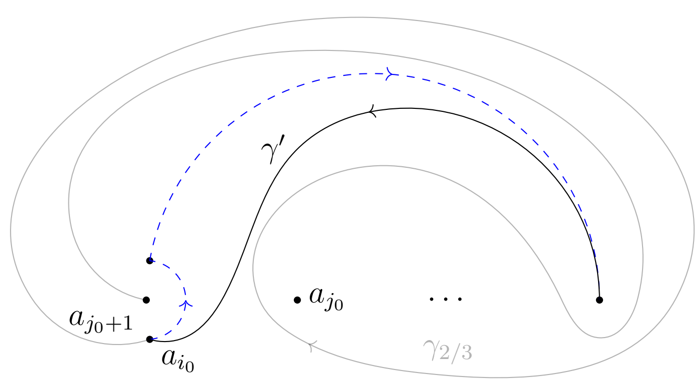
To justify (5.1.3), first we deform γ_0 into $\gamma_{1/3}$. As shown in Figure 5.1.4a. The faint path is γ_0, while the dashed path is the trace of a_{i_0}.



(a) Deformation from $I_{\gamma_0}(a_{i_0};\cdots;a_{j_0+1})$ to $I_{\gamma_{1/3}}(a_{i_0};\cdots;a_{j_0+1})$



(b) Deformation from $I_{\gamma_{1/3}}(a_{i_0};\cdots;a_{j_0+1})$ to $I_{\gamma_{2/3}}(a_{i_0};\cdots;a_{j_0+1})$

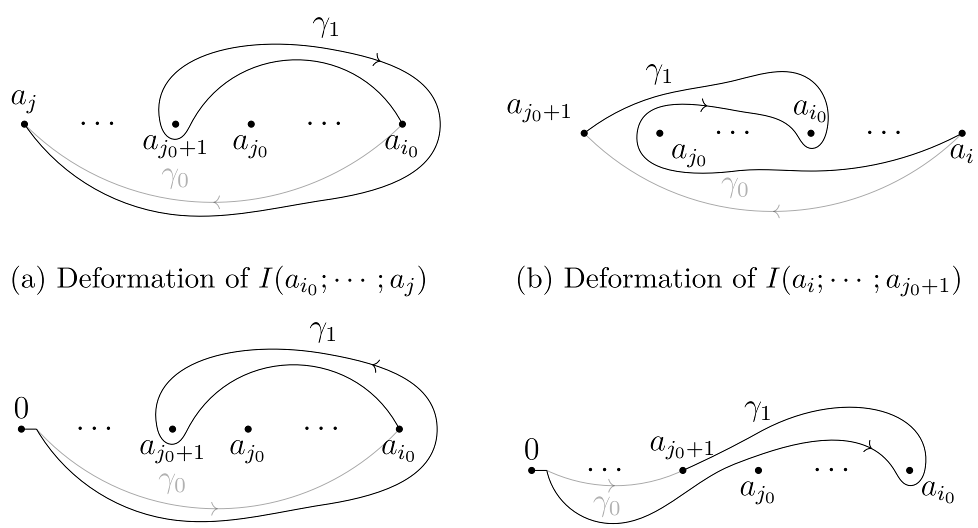


(c) Deformation from $I_{\gamma_{2/3}}(a_{i_0};\cdots;a_{j_0+1})$ to $I_{\gamma_1}(a_{i_0};\cdots;a_{j_0+1})$

Figure 5.1.4: Deformation of $I(a_{i_0};\cdots;a_{j_0+1})$

Notice that we didn't take the traces of $a_1, \cdots, a_{i_0-1}$ into account, simply because the iterated integral $I(a_{i_0;\cdots;a_{j_0+1}})$ is not affected by them. Next, we deform $\gamma_{1/3}$ into $\gamma_{2/3}$. This is illustrated in Figure 5.1.4b. Lastly, we can stretch $\gamma_{2/3}$ into $\gamma_1 = \gamma' \gamma_{2/3}$. This is presented in Figure 5.1.4c. This proves the first equality in (5.1.3). The second equality holds since a_{i_0} and a_{j_0+1} are the endpoints, so the monodromy σ_{i_0}, σ_{j_0+1} are trivial. The last equality holds because σ's cancel off each other.

The deformations in (5.1.4), (5.1.5), (5.1.6) and (5.1.7) are similar. The eventual paths γ_1 are presented in Figure 5.1.5.



(a) Deformation of $I(a_{i_0}; \cdots; a_j)$

(b) Deformation of $I(a_i; \cdots; a_{j_0+1})$

(c) Deformation of $I(0; \cdots; a_{i_0})$

(d) Deformation of $I(0; \cdots; a_{j_0+1})$

Figure 5.1.5: Deformations for (5.1.4), (5.1.5), (5.1.6) and (5.1.7)

$\square$

5.2 Computation of monodromy matrices

5.2.1 Monodromies of iterated integrals

In this section, let us assume $\{a_i\}_i \subseteq \mathbb{C}$, and σ_p are the loops such that $\displaystyle\int_{\sigma_q} d\log(z-a_q) = 2\pi i \delta_{pq}$ where δ is the Kronecker delta. The following Lemma is the key to the calculation of the monodromies matrices.

Lemma 5.2.1. (Corollary 2.6 in [21], Proposition 6.3 in [19]) Suppose $\gamma = \gamma_1'\gamma_2'$ is a path from a_0 to a_{n+1} and $\gamma' = \gamma_1'\sigma\gamma_2'$, $a_i \neq a$ for $1 \leq i \leq n$, then

$$I_{\gamma'}(a_0;\cdots,\overbrace{a,\cdots,a}^{k},\cdots;a_{n+1}) - I_{\gamma}(a_0;\cdots,\overbrace{a,\cdots,a}^{k},\cdots;a_{n+1}) \tag{5.2.1}$$

is equal to

$$\sum_{\substack{p+q+r=k\\ r\geq 1}} I_{\gamma_1}(a_0;\cdots,\overbrace{a,\cdots,a}^{p};a)I_{\sigma}(a;\overbrace{a,\cdots,a}^{r};a)I_{\gamma_2}(a;\overbrace{a,\cdots,a}^{q},\cdots;a_{n+1}) \tag{5.2.2}$$

Which is equal to

$$\sum_{\substack{p+q+r=k\\ r\geq 1}} \frac{(2\pi i)^r}{(r)!} I_{\gamma_1}(a_0;\cdots,\overbrace{a,\cdots,a}^{p};a)I_{\gamma_2}(a;\overbrace{a,\cdots,a}^{q},\cdots;a_{n+1}) \tag{5.2.3}$$

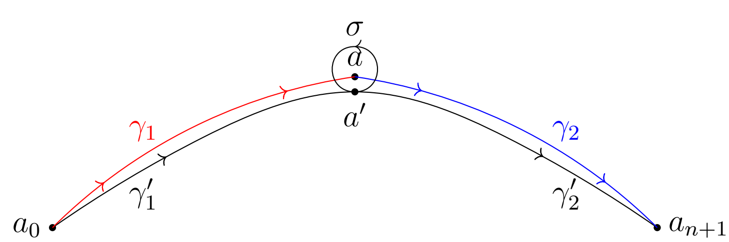


Figure 5.2.1: Monodromy of $I(a_0;\cdots,a,\cdots,a,\cdots;a_{n+1})$ at a

Proof. Let's write $\omega_p = \dfrac{dt}{t-a_p}$ and $\omega = \dfrac{dt}{t-a}$, then (5.2.3) can be written as

$$\begin{aligned}
&\int_{\gamma_1'\sigma\gamma_2'} \omega_1\cdots\overbrace{\omega\cdots\omega}^{k}\cdots\omega_n - \int_{\gamma_1'\gamma_2'} \omega_1\cdots\overbrace{\omega\cdots\omega}^{k}\cdots\omega_n \\
&= \sum_{\substack{p+q+r=k\\ r\geq 1}} \int_{\gamma_1'} \omega_1\cdots\overbrace{\omega\cdots\omega}^{p} \int_{\sigma} \overbrace{\omega\cdots\omega}^{r} \int_{\gamma_2'} \overbrace{\omega\cdots\omega}^{q}\cdots\omega_n \\
&+ \sum_{i}\sum_{q+r=k} \int_{\gamma_1'} \omega_1\cdots\omega_i \int_{\sigma} \omega_{i+1}\cdots\overbrace{\omega\cdots\omega}^{r} \int_{\gamma_2'} \overbrace{\omega\cdots\omega}^{q}\cdots\omega_n \\
&+ \sum_{j}\sum_{p+r=k} \int_{\gamma_1'} \omega_1\cdots\overbrace{\omega\cdots\omega}^{p} \int_{\sigma} \overbrace{\omega\cdots\omega}^{r}\cdots\omega_j \int_{\gamma_2'} \omega_{j+1}\cdots\omega_n
\end{aligned}$$

When a' approaches a, we may take the limit of γ_1', γ_2' to be γ_1, γ_2 respectively. If we choose σ to be $a+\epsilon e^{i\theta}$, and let $\epsilon \to 0$ we have

$$\int_{\sigma} \overbrace{\omega\cdots\omega}^{r} = \int_0^{2\pi} \overbrace{\frac{\epsilon i e^{i\theta}d\theta}{\epsilon e^{i\theta}}\cdots\frac{\epsilon i e^{i\theta}d\theta}{\epsilon e^{i\theta}}}^{r} = i^r \int_0^{2\pi} \overbrace{d\theta\cdots d\theta}^{r} = \frac{(2\pi i)^r}{r!}$$

which explains the coefficients in the first sum. To prove that the second and third sum vanish as $\epsilon \to 0$, note that

$$\begin{aligned}
\left|\int_{\sigma} \omega_{i+1}\cdots\overbrace{\omega\cdots\omega}^{r}\right| &= \left|\int_0^{2\pi} \frac{\epsilon i e^{i\theta}d\theta}{a-a_{i+1}+\epsilon e^{i\theta}}\cdots\overbrace{\frac{\epsilon i e^{i\theta}d\theta}{\epsilon e^{i\theta}}\cdots\frac{\epsilon i e^{i\theta}d\theta}{\epsilon e^{i\theta}}}^{r}\right| \\
&\leq \epsilon\cdot\frac{1}{|a-a_{i+1}|-\epsilon}\cdots\left|\int_0^{2\pi} \overbrace{\frac{\epsilon i e^{i\theta}d\theta}{\epsilon e^{i\theta}}\cdots\frac{\epsilon i e^{i\theta}d\theta}{\epsilon e^{i\theta}}}^{r}\right| \leq \epsilon\cdot C
\end{aligned}$$

and similarly $\left|\int_{\sigma} \overbrace{\omega\cdots\omega}^{r}\cdots\omega_j\right| \leq \epsilon\cdot C$. One might argue that $\int_{\gamma_2'} \overbrace{\omega\cdots\omega}^{q}\cdots\omega_n$ is singular, but we know from Proposition 2.1.19 that it is $O(\log^q \epsilon)$. Therefore we have proved (5.2.3). □

Remark 5.2.2. If we replace σ with σ^{-1}, then

$$\int_{\sigma^{-1}} \overbrace{\omega\cdots\omega}^{r} = (-1)^r \int_{\sigma} \overbrace{\omega\cdots\omega}^{r} = \frac{(-2\pi i)^r}{r!}$$

and (5.2.2) becomes

$$\sum_{\substack{p+q+r=k \\ r\geq 1}} \frac{(-2\pi i)^r}{(r)!} I_{\gamma_1}(a_0;\cdots,\overbrace{a,\cdots,a}^{p};a) I_{\gamma_2}(a;\overbrace{a,\cdots,a}^{q},\cdots;a_{n+1}) \tag{5.2.4}$$

When γ, γ_1, γ_2 are fixed choices, without ambiguity, we omit the mention of the integration path γ, and write only its monodromy loops. For instance, we simplify $I_{\gamma'}$ to I_σ and I_γ, I_{γ_1}, I_{γ_2} to I. In addition, for degenerates $I(a;\cdots;a)$, we pick the trivial path so that they evaluate to zero.

If we remove the condition $a_i \neq a$, for $1 \leq i \leq n$, and deploy the notation in Definition 4.1.11. Lemma 5.2.1 can be easily generalized.

Corollary 5.2.3.

$$I_{\sigma_p^\epsilon}(a_{i_0};a_{i_1},\cdots,a_{i_n};a_{i_{n+1}}) = \sum_{k=0}^{\infty} \frac{(2\pi i\epsilon)^k}{k!} I^{\sigma_p^k}(a_{i_0};a_{i_1},\cdots,a_{i_n};a_{i_{n+1}}) \tag{5.2.5}$$

Here $\epsilon = \pm 1$ is the sign, $I(a;a)=1$ and degenerates vanish. For a product of loops, we have

$$I_{\sigma_{p_1}^{\epsilon_1}\cdots\sigma_{p_m}^{\epsilon_m}}(a_{i_0};\cdots;a_{i_{n+1}}) = \sum_{k_1,\cdots,k_l\geq 0} \prod_{r=1}^{m} \frac{(2\pi i\epsilon_r)^{k_r}}{k_r!} I^{\sigma_{p_1}^{k_1}\cdots\sigma_{p_m}^{k_m}}(a_{i_0};\cdots;a_{i_{n+1}}) \tag{5.2.6}$$

Where $\epsilon_r = \pm 1$. By Proposition 2.1.6, ii., We also deduce that

$$\begin{aligned}
I_{\sigma_{p_m}^{\epsilon_m}\cdots\sigma_{p_1}^{\epsilon_1}}(a_{i_{n+1}};\cdots;a_{i_0}) &= \sum_{k_1,\cdots,k_l\geq 0} \prod_{r=1}^{m} \frac{(2\pi i\epsilon_r)^{k_r}}{k_r!} I^{\sigma_{p_m}^{k_m}\cdots\sigma_{p_1}^{k_1}}(a_{i_{n+1}};\cdots;a_{i_0}) \\
&= \sum_{k_1,\cdots,k_l\geq 0} \prod_{r=1}^{m} \frac{(2\pi i\epsilon_r)^{k_r}}{k_r!} (-1)^{n-k_m-\cdots-k_1} I^{\sigma_{p_1}^{k_1}\cdots\sigma_{p_m}^{k_m}}(a_{i_0};\cdots;a_{i_{n+1}}) \\
&= (-1)^n \sum_{k_1,\cdots,k_l\geq 0} \prod_{r=1}^{m} \frac{(-2\pi i\epsilon_r)^{k_r}}{k_r!} I^{\sigma_{p_1}^{k_1}\cdots\sigma_{p_m}^{k_m}}(a_{i_0};\cdots;a_{i_{n+1}}) \\
&= (-1)^n I_{\sigma_{p_1}^{-\epsilon_1}\cdots\sigma_{p_m}^{-\epsilon_m}}(a_{i_0};\cdots;a_{i_{n+1}})
\end{aligned} \tag{5.2.7}$$

Remark 5.2.4. If $p_r = p_{r+1}$ and $\epsilon_r = -\epsilon_{r+1}$, then $\sigma_{p_r}^{\epsilon_r}\sigma_{p_{r+1}}^{\epsilon_{r+1}} = 1$ may cancel each other.

Proof. This is just applying Lemma 5.2.1 repeatedly. $\square$

5.2.2 Computation of monodromy matrices

Recall from Proposition 4.1.13, that

$$(-1)^{l-k} I^{\sigma_{i_1}\sigma_0^{m_{i_1}-1}\cdots\sigma_{i_k}\sigma_0^{m_{i_k}-1}}(0; a_{j_1}, 0^{p_{j_1}-1}, \cdots, a_{j_l}, 0^{p_{j_l}-1}; 1) \tag{5.2.8}$$

is the complementary entry of $(-1)^k I(0; a_{i_1}, 0^{m_{i_1}-1}, \cdots, a_{i_k}, 0^{m_{i_k}-1}; 1)$ with respect to $(-1)^l I(0; a_{j_1}, 0^{p_{j_1}-1}, \cdots, a_{j_l}, 0^{p_{j_l}-1}; 1)$. Now we can describe a concrete algorithm for computing monodromy matrices. For this, we only need to compute the entry corresponding to (5.2.8) in the monodromy matrix.

First we discuss the monodromy matrix for $\mathcal{M}_{\nu_{i_0}}$.

Theorem 5.2.5. Suppose $i_r \le i_0 < i_{r+1}$, and denote $w_0 = \sigma_{i_1}\sigma_0^{m_{i_1}-1}\cdots\sigma_{i_k}\sigma_0^{m_{i_k}-1}$, we have

$$\mathcal{M}_{\nu_{i_0}} I^{w_0}(0; a_{j_1}, 0^{p_{j_1}-1}, \cdots, a_{j_l}, 0^{p_{j_l}-1}; 1) = \sum_w M_{w,w_0}^{(i_0)} I^w(0; a_{j_1}, 0^{p_{j_1}-1}, \cdots, a_{j_l}, 0^{p_{j_l}-1}; 1) \tag{5.2.9}$$

With constant coefficient

$$M_{w,w_0}^{(i_0)} = \sum_{\substack{\theta_\alpha \in \{0,1\} \\ q_0, q_r \ge 0}} \frac{(-2\pi i)^{q_0}}{q_0!}(2\pi i)^{\theta_{i_0+1}+\cdots+\theta_{i_{r+1}-1}} \frac{(2\pi i)^{q_r}}{q_r!} \tag{5.2.10}$$

for $w = \sigma_0^{q_0}\sigma_{i_1}\sigma_0^{m_{i_1}-1}\cdots\sigma_{i_r}\sigma_{i_0+1}^{\theta_{i_0+1}}\cdots\sigma_{i_{r+1}-1}^{\theta_{i_{r+1}-1}}\sigma_0^{m_{i_r}-1+q_r}\cdots\sigma_{i_k}\sigma_0^{m_{i_k}-1}$, and otherwise zero. It is then straightfoward to see that $M^{(i_0)} = \left\{(-1)^{k+\theta_{i_0+1}+\cdots+\theta_{i_{r+1}-1}} M_{w,v}^{(i_0)}\right\}_{w,v}$ defines precisely the monodromy matrix for the operator $\mathcal{M}_{\nu_{i_0}}$.

Proof. First notice that

$$I^{\sigma_{i_1}\sigma_0^{m_{i_1}-1}\cdots\sigma_{i_k}\sigma_0^{m_{i_k}-1}}(0;a_{j_1},0^{p_{j_1}-1},\cdots,a_{j_l},0^{p_{j_l}-1};1)$$
$$=I(0;\cdots;a_{i_1})\left(\prod_{t=1}^{k}I^{\sigma_0^{m_{i_t}-1}}(a_{i_t};\cdots;a_{i_{t+1}})\right)\quad(5.2.11)$$

And if $m_{i_t}>1$,

$$I^{\sigma_0^{m_{i_t}-1}}(a_{i_t};\cdots;a_{i_{t+1}})=\sum I(a_{i_t};\cdots;0)I(0;\cdots;a_{i_{t+1}})\quad(5.2.12)$$

Thanks to Corollary 5.2.3 and Theorem 5.1.1, we have

$$\mathcal{M}_{\nu_{i_0}}I(0;\cdots;a_{i_1})=\sum_{q_0\geq 0}\frac{(-2\pi i)^{q_0}}{q_0!}I^{\sigma_0^{q_0}}(0;\cdots;a_{i_1})\quad(5.2.13)$$

If $m_{i_r}=1$,

$$\mathcal{M}_{\nu_{i_0}}I(a_{i_r};\cdots;a_{i_{r+1}})$$
$$=\sum_{\substack{\theta_{i_0+1},\cdots,\theta_{i_{r+1}-1}\in\{0,1\}\\ q_r\geq 0}}(2\pi i)^{\theta_{i_0+1}+\cdots+\theta_{i_{r+1}-1}}\frac{(2\pi i)^{q_r}}{q_r!}I^{\sigma_{i_0+1}^{\theta_{i_0+1}}\cdots\sigma_{i_{r+1}-1}^{\theta_{i_{r+1}-1}}\sigma_0^{q_r}}(a_{i_r};\cdots;a_{i_{r+1}})$$
$$(5.2.14)$$

And if $m_{i_r}>1$,

$$\mathcal{M}_{\nu_{i_0}}I^{\sigma_0^{m_{i_r}-1}}(a_{i_r};\cdots;a_{i_{r+1}})=\sum\mathcal{M}_{\nu_{i_0}}I(a_{i_r};\cdots;0)\mathcal{M}_{\nu_{i_0}}I(0;\cdots;a_{i_{r+1}})$$
$$=\sum_{\substack{\theta_\alpha\in\{0,1\}\\ q_r\geq 0}}(2\pi i)^{\theta_{i_0+1}+\cdots+\theta_{i_{r+1}-1}}\frac{(2\pi i)^{q_r}}{q_r!}I^{\sigma_{i_0+1}^{\theta_{i_0+1}}\cdots\sigma_{i_{r+1}-1}^{\theta_{i_{r+1}-1}}\sigma_0^{q_r}}(a_{i_r};\cdots;0)I(0;\cdots;a_{i_{r+1}})$$
$$=\sum_{\substack{\theta_\alpha\in\{0,1\}\\ q_r\geq 0}}(2\pi i)^{\theta_{i_0+1}+\cdots+\theta_{i_{r+1}-1}}\frac{(2\pi i)^{q_r}}{q_r!}I^{\sigma_{i_0+1}^{\theta_{i_0+1}}\cdots\sigma_{i_{r+1}-1}^{\theta_{i_{r+1}-1}}\sigma_0^{m_{i_r}-1+q_r}}(a_{i_r};\cdots;a_{i_{r+1}})$$
$$(5.2.15)$$

Note that for the second equality (5.2.7) is used. If $m_{i_t} > 1$ and $t < r$,

$$\begin{aligned}
\mathcal{M}_{\nu_{i_0}} I^{\sigma_0^{m_{i_t}-1}}(a_{i_t}; \cdots; a_{i_{t+1}}) &= \sum \mathcal{M}_{\nu_{i_0}} I(a_{i_t}; \cdots; 0) \mathcal{M}_{\nu_{i_0}} I(0; \cdots; a_{i_{t+1}}) \\
&= \sum_{x,y\geq 0} \frac{(2\pi i)^x}{x!} I^{\sigma_0^x}(a_{i_t}; \cdots; 0) \frac{(-2\pi i)^y}{y!} I^{\sigma_0^y}(0; \cdots; a_{i_{t+1}}) \\
&= \sum_{q_t \geq 0} (2\pi i)^{q_t} \sum_{x+y=q_t} \frac{(-1)^y}{x!y!} I^{\sigma_0^x}(a_{i_t}; \cdots; 0) I^{\sigma_0^y}(0; \cdots; a_{i_{t+1}}) \\
&= 0
\end{aligned} \tag{5.2.16}$$

To summarize, we have

$$\begin{aligned}
&\mathcal{M}_{\nu_{i_0}} I^{\sigma_{i_1}\sigma_0^{m_{i_1}-1}\cdots\sigma_{i_k}\sigma_0^{m_{i_k}-1}}(0; a_{j_1}, 0^{p_{j_1}-1}, \cdots, a_{j_l}, 0^{p_{j_l}-1}; 1) \\
&\quad = \sum_{\substack{\theta_\alpha \in \{0,1\} \\ q_0, q_r \geq 0}} \frac{(-2\pi i)^{q_0}}{q_0!} (2\pi i)^{\theta_{i_0+1}+\cdots+\theta_{i_{r+1}-1}} \frac{(2\pi i)^{q_r}}{q_r!} \\
&\quad I^{\sigma_0^{q_0}\sigma_{i_1}\sigma_0^{m_{i_1}-1}\cdots\sigma_{i_r}\sigma_{i_0+1}^{\theta_{i_0+1}}\cdots\sigma_{i_{r+1}-1}^{\theta_{i_{r+1}-1}}\sigma_0^{m_{i_r}-1+q_r}\cdots\sigma_{i_k}\sigma_0^{m_{i_k}-1}}(0; a_{j_1}, 0^{p_{j_1}-1}, \cdots, a_{j_l}, 0^{p_{j_l}-1}; 1)
\end{aligned} \tag{5.2.17}$$

□

Next we discuss the monodromy matrix for $\mathcal{M}_{\nu_{i_0,j_0}}$.

Theorem 5.2.6.

$$\mathcal{M}_{\nu_{i_0,j_0}} I^{w_0}(0; a_{j_1}, 0^{p_{j_1}-1}, \cdots, a_{j_l}, 0^{p_{j_l}-1}; 1) = \sum_w M_{w,w_0}^{(i_0,j_0)} I^w(0; a_{j_1}, 0^{p_{j_1}-1}, \cdots, a_{j_l}, 0^{p_{j_l}-1}; 1) \tag{5.2.18}$$

With constant coefficient $M_{w,w_0}^{(i_0,j_0)}$ being

$$(-1)^\delta (2\pi i)^{\delta(\theta_{i_0+1}+\cdots+\theta_{j_0}+1)} \tag{5.2.19}$$

for $w = \sigma_{i_1}\sigma_0^{m_{i_1}-1}\cdots\sigma_{i_r}\sigma_{i_0+1}^{\delta\theta_{i_0+1}}\cdots\sigma_{j_0}^{\delta\theta_{j_0}}\sigma_{j_0+1}^{\delta}\sigma_0^{m_{i_r}-1}\cdots\sigma_{i_k}\sigma_0^{m_{i_k}-1}$, $i_r = i_0 < j_0 + 1 < i_{r+1}$, and

$$(2\pi i)^{\delta(1+\theta_{i_0+1}+\cdots+\theta_{j_0})} \tag{5.2.20}$$

for $w = \sigma_{i_1}\sigma_0^{m_{i_1}-1}\cdots\sigma_{i_r}\sigma_0^{m_{i_r}-1}\sigma_{i_0}^{\delta}\sigma_{i_0+1}^{\delta\theta_{i_0+1}}\cdots\sigma_{j_0}^{\delta\theta_{j_0}}\cdots\sigma_{i_k}\sigma_0^{m_{i_k}-1}$, $i_r < i_0 < j_0+1 = i_{r+1}$. It is then straightforward to see that $M^{(i_0,j_0)} = \left\{(-1)^{k+\delta(1+\theta_{i_0+1}+\cdots+\theta_{j_0})}M_{w,v}^{(i_0,j_0)}\right\}_{w,v}$ defines precisely the monodromy matrix for the operator $\mathcal{M}_{\nu_{i_0,j_0}}$.

Proof. Again with the help of Corollary 5.2.3 and Theorem 5.1.2, it is not difficult to show that

$$\mathcal{M}_{\nu_{i_0,j_0}} I(a_{i_0};\cdots;a_j) = \sum_{\theta_\alpha,\delta\in\{0,1\}} (-1)^{\delta}(2\pi i)^{\delta(\theta_{i_0+1}+\cdots+\theta_{j_0}+1)} I^{\sigma_{i_0+1}^{\delta\theta_{i_0+1}}\cdots\sigma_{j_0}^{\delta\theta_{j_0}}\sigma_{j_0+1}^{\delta}}(a_{i_0};\cdots;a_j) \tag{5.2.21}$$

$$\mathcal{M}_{\nu_{i_0,j_0}} I(a_{i_0};\cdots;0) = \sum_{\theta_\alpha,\delta\in\{0,1\}} (-1)^{\delta}(-2\pi i)^{\delta(\theta_{i_0+1}+\cdots+\theta_{j_0}+1)} I^{\sigma_{i_0+1}^{\delta\theta_{i_0+1}}\cdots\sigma_{j_0}^{\delta\theta_{j_0}}\sigma_{j_0+1}^{\delta}}(a_{i_0};\cdots;0) \tag{5.2.22}$$

$$\mathcal{M}_{\nu_{i_0,j_0}} I(a_i;\cdots;a_{j_0+1}) = \sum_{\theta_\alpha,\delta\in\{0,1\}} (2\pi i)^{\delta(1+\theta_{i_0+1}+\cdots+\theta_{j_0})} I^{\sigma_{i_0}^{\delta}\sigma_{i_0+1}^{\delta\theta_{i_0+1}}\cdots\sigma_{j_0}^{\delta\theta_{j_0}}}(a_i;\cdots;a_{j_0+1}) \tag{5.2.23}$$

$$\mathcal{M}_{\nu_{i_0,j_0}} I(0;\cdots;a_{j_0+1}) = \sum_{\theta_\alpha,\delta\in\{0,1\}} (2\pi i)^{\delta(1+\theta_{i_0+1}+\cdots+\theta_{j_0})} I^{\sigma_{i_0}^{\delta}\sigma_{i_0+1}^{\delta\theta_{i_0+1}}\cdots\sigma_{j_0}^{\delta\theta_{j_0}}}(0;\cdots;a_{j_0+1}) \tag{5.2.24}$$

If $i_0 \le i_r < i_{r+1} \le j_0+1$ or $\{i_0, j_0+1\}\cap\{i_r\}_{r=1}^{k} = \emptyset$, $\mathcal{M}_{\nu_{i_0,j_0}}$ acts trivially.

If $i_r = i_0 < j_0+1 < i_{r+1}$,

$$\begin{aligned}&\mathcal{M}_{\nu_{i_0,j_0}} I^{\sigma_0^{m_{i_r}-1}}(a_{i_0};\cdots;a_{i_{r+1}}) \\ &= \sum_{\theta_\alpha,\delta\in\{0,1\}} (-1)^{\delta}(2\pi i)^{\delta(\theta_{i_0+1}+\cdots+\theta_{j_0}+1)} I^{\sigma_{i_0+1}^{\delta\theta_{i_0+1}}\cdots\sigma_{j_0}^{\delta\theta_{j_0}}\sigma_{j_0+1}^{\delta}\sigma_0^{m_{i_r}-1}}(a_{i_0};\cdots;a_{i_{r+1}})\end{aligned} \tag{5.2.25}$$

therefore

$$\begin{aligned}&\mathcal{M}_{\nu_{i_0,j_0}} I^{\sigma_{i_1}\sigma_0^{m_{i_1}-1}\cdots\sigma_{i_k}\sigma_0^{m_{i_k}-1}}(0;a_{j_1},0^{p_{j_1}-1},\cdots,a_{j_l},0^{p_{j_l}-1};1) \\ &= \sum_{\theta_\alpha,\delta\in\{0,1\}} (-1)^{\delta}(2\pi i)^{\delta(\theta_{i_0+1}+\cdots+\theta_{j_0}+1)} \\ &I^{\sigma_{i_1}\sigma_0^{m_{i_1}-1}\cdots\sigma_{i_r}\sigma_{i_0+1}^{\delta\theta_{i_0+1}}\cdots\sigma_{j_0}^{\delta\theta_{j_0}}\sigma_{j_0+1}^{\delta}\sigma_0^{m_{i_r}-1}\cdots\sigma_{i_k}\sigma_0^{m_{i_k}-1}}(0;a_{j_1},0^{p_{j_1}-1},\cdots,a_{j_l},0^{p_{j_l}-1};1)\end{aligned} \tag{5.2.26}$$

If $i_r < i_0 < j_0 + 1 = i_{r+1}$,

$$\begin{aligned}
&\mathcal{M}_{\nu_{i_0,j_0}} I^{\sigma_0^{m_{i_r}-1}}(a_{i_r};\cdots;a_{j_0+1}) \\
&\qquad = \sum_{\theta_\alpha,\delta\in\{0,1\}} (2\pi i)^{\delta(1+\theta_{i_0+1}+\cdots+\theta_{j_0})} I^{\sigma_0^{m_{i_r}-1}\sigma_{i_0}^{\delta}\sigma_{i_0+1}^{\delta\theta_{i_0+1}}\cdots\sigma_{j_0}^{\delta\theta_{j_0}}}(a_{i_r};\cdots;a_{j_0+1}) \quad (5.2.27)
\end{aligned}$$

therefore,

$$\begin{aligned}
&\mathcal{M}_{\nu_{i_0,j_0}} I^{\sigma_{i_1}\sigma_0^{m_{i_1}-1}\cdots\sigma_{i_k}\sigma_0^{m_{i_k}-1}}(0;a_{j_1},0^{p_{j_1}-1},\cdots,a_{j_l},0^{p_{j_l}-1};1) \\
&= \sum_{\theta_\alpha,\delta\in\{0,1\}} (2\pi i)^{\delta(1+\theta_{i_0+1}+\cdots+\theta_{j_0})} \quad (5.2.28) \\
&I^{\sigma_{i_1}\sigma_0^{m_{i_1}-1}\cdots\sigma_{i_r}\sigma_0^{m_{i_r}-1}\sigma_{i_0}^{\delta}\sigma_{i_0+1}^{\delta\theta_{i_0+1}}\cdots\sigma_{j_0}^{\delta\theta_{j_0}}\cdots\sigma_{i_k}\sigma_0^{m_{i_k}-1}}(0;a_{j_1},0^{p_{j_1}-1},\cdots,a_{j_l},0^{p_{j_l}-1};1)
\end{aligned}$$

□

Example 5.2.7. The monodromy matrix of $\mathrm{Li}_{3,1}(x_1,x_2)$ for ν_1, ν_2, $\nu_{1,1}$, $\nu_{2,2}$, $\nu_{1,2}$ are respectively

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_{1,1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_{2,2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$M_{1,2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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